

Exotic smooth \mathbb{R}^4 , geometry of string backgrounds and quantum D-branes

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Abstract

In this paper we make a first step toward determining 4-dimensional data from higher dimensional superstring theory and considering these as underlying structures for the theory. First, we explore connections of exotic smoothings of \mathbb{R}^4 and certain configurations of NS and D-branes, both classical and (generalized) quantum using C^* algebras. Effects of some small exotic \mathbb{R}^4 's, when localized on $S^3 \subset \mathbb{R}^4$, correspond to stringy geometries of B -fields on S^3 . Exotic smoothness of \mathbb{R}^4 acts as a non-vanishing B -field on S^3 in \mathbb{R}^4 . The dynamics of D-branes in $SU(2)$ WZW model at finite k indicates the exoticness of ambient \mathbb{R}^4 .

Next, based on the relation of exotic smooth \mathbb{R}^4 with integral levels of $SU(2)$ WZW model we show the correspondence between exotic smoothness on 4-space, transversal to the world volume of NS5 branes, and the number of these NS5 branes. Relation with the calculations in holographically dual 6-dimensional little string theory is discussed.

Generalized quantum D-branes in the noncommutative C^* algebras corresponding to the codimension-1 foliations of S^3 are considered and these determine the KK invariants of exotic smooth \mathbb{R}_h^4 for the case of non-integral $[h] \in H^3(S^3, \mathbb{R})$. Moreover, exotic smooth \mathbb{R}^4 's embedded in some exotic \mathbb{R}^4 as open submanifolds, are shown to correspond to generalized quantum D-branes in the noncommutative C^* algebra of the foliation. Finally, we show how exotic smoothness of \mathbb{R}^4 is correlated with D6 brane charges in IIA string theory.

In the last section we construct wild embeddings of spheres and relate them to D-brane charges as well to KK theory. Wild embeddings as constructed by using gropes are basic to understand exotic smoothness as well Casson handles. Finally we conjecture that a quantum D-brane is wild embedding. Then we construct an action for a quantum D-brane and show that the classical limit (the usual embedding) agrees with the Born-Infeld action.

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1. Introduction

Despite the substantial effort toward quantizing gravity in 4 dimensions, this issue is still open. One of the best candidates till now is the superstring theory formulated in 10 dimensions. A way from superstring theory to 4-dimensional quantum gravity or standard model of particle physics (minimal supersymmetric extension thereof) is, at best, highly non-unique. Many techniques of compactifications and flux stabilization along with specific model-building branes configurations and dualities, were worked out toward this end within the years. Possibly some important data of a fundamental character are still missing.

The point of view advocated in this paper is that indeed we have not respected till now 4-dimensional phenomena of different smoothings of Euclidean \mathbb{R}^4 which presumably are very important for the program of QG. There are strong evidences that exotic 4-smoothness on compact manifolds should be taken into account by any QG theory [7]. Here we refer to open 4-manifolds and try to consider exotic \mathbb{R}^4 's as serving a link between higher dimensional superstring theory and 4-dimensional „physical“ theories and 4-dimensional QG.

String theory would describe directly 4-dimensional structures at the fundamental level. This paper serves as a step toward seeing exotic smooth \mathbb{R}^4 as fundamental objects underlying higher dimensional (super)string theory. Further results regarding compactification and realistic 4-dimensional models of various brane configurations in string theory and their relation to exotic 4-smoothings, will be presented separately.

The problem with successful inclusion of the effects of 4-open-exotics into any physical theory, is the notorious lack of an explicit coordinate-like description of these smooth manifolds. In the series of our recent papers we addressed this issue and worked out some relative techniques allowing for analytical treatment of small exotic \mathbb{R}^4 's [10, 9, 8, 33]. In this paper we show that description of D-branes in some exact string backgrounds are related with 4-smoothness of \mathbb{R}^4 . Moreover, the deep quantum regime of the D-branes is also 4-exotic sensitive.

However the connection of abstract, generalized quantum D-branes to the actual superstring theory D-branes (in the manifold limit) is not directly given. The Witten limit of superstring theory where D-branes yield their noncommutative world-volumes is only the midway and in fact motivates the full C^* algebra approach [15]. This last serves as a possible partial solution to the problem of describing quantum D-branes in superstring theory. The connection with exotic \mathbb{R}^4 at this, quantum level is unexpected and shows that 4-dimensionality may get into the game in string theory through back-door of nonperturbative quantum regime. In the last section of the paper we use the C^* algebra approach to quantum D-branes to construct a manifold model of a quantum D-brane as wild embedding. Then we show that the C^* algebra of the wild embedding is isomorphic to the C^* algebra of the quantum D-brane. Furthermore we construct a quantum version of an action using cyclic cohomology and get the right limit to the classical D-brane described by the Born-Infeld action.

The basic technical ingredient of the analysis of small exotic \mathbb{R}^4 's enabling uncovering many applications also in string theory is the relation between exotic (small) \mathbb{R}^4 's and non-cobordant codimension-1 foliations of the S^3 as well gropes and wild embeddings as shown in [10]. The foliation are classified by Godbillon-Vey class as element of the cohomology group $H^3(S^3, \mathbb{R})$. By using the S^1 -gerbes it was possible to interpret the integral elements $H^3(S^3, \mathbb{Z})$ as characteristic classes of a S^1 -gerbe over S^3 [9].

The main line of the topological argumentation can be briefly described as follows:

1. In Bizacas exotic \mathbb{R}^4 one starts with the neighborhood $N(A)$ of the Akbulut cork A in the K3 surface M . The exotic \mathbb{R}^4 is the interior of $N(A)$.
2. This neighborhood $N(A)$ decomposes into A and a Casson handle representing the non-trivial involution of the cork.
3. From the Casson handle we construct a grope containing Alexanders horned sphere.
4. Akbuluts construction gives a non-trivial involution, i.e. the double of that construction is the identity map.
5. From the grope we get a polygon in the hyperbolic space \mathbb{H}^2 .

6. This polygon defines a codimension-1 foliation of the 3-sphere inside of the exotic \mathbb{R}^4 with an wildly embedded 2-sphere, Alexanders horned sphere.
7. Finally we get a relation between codimension-1 foliations of the 3-sphere and exotic \mathbb{R}^4 .

This relation is very strict, i.e. if we change the Casson handle then we must change the polygon. But that changes the foliation and vice versa. Finally we obtained the result:

The exotic \mathbb{R}^4 (of Bizaca) is determined by the codimension-1 foliations with non-vanishing Godbillon-Vey class in $H^3(S^3, \mathbb{R}^3)$ of a 3-sphere seen as submanifold $S^3 \subset \mathbb{R}^4$.

2. Geometry of string backgrounds and exotic \mathbb{R}^4

In this section we take the point of view that exotic smoothness of some small exotic \mathbb{R}^4 's when localized on $S^3 \subset \mathbb{R}^4$, correspond to some stringy geometry given by so-called B -fields on S^3 . The localization is understood as the representation of the exotics by 3-rd integral or real cohomologies of S^3 . This correspondence takes place in fact for a classical limit of the geometry of string backgrounds, i.e. curved Riemannian manifold with B-field. One can say that localized exotic smooth \mathbb{R}^4 on S^3 is described by stringy geometry of B -fields on this S^3 . The correspondence can be extended over string regime of finite volume of $SU(2)$ WZW model.

2.1. $SU(2)$ WZW model, D-branes and exotic \mathbb{R}^4

We want to focus on changing the smoothness of \mathbb{R}^4 and considering the changes as localized on S^3 . As follows from [10, 9] this gives rise to stringy effects, since the changes can be described by computations in some 2D CFT, namely WZW models on $SU(2)$ at finite level.

First we are going to discuss bosonic $SU(2)$ WZW model and dynamics of branes in it. We deal here with S^3 hence the nonzero metric of string background. In general, non-vanishing curvature $R(g)$, where g is a non-constant metric, of the background manifold (M, g) on which bosonic string theory is formulated, enforces that H -field on M cannot vanish. This is since the string field equations gives rise to (see e.g. [37])

$$R_{\mu\nu}(g) - \frac{1}{4}H_{\mu\rho\sigma}H_{\nu}{}^{\rho\sigma} = \mathcal{O}(\alpha') \quad (1)$$

where $H = dB$ is the NSNS 3-form, $B = B(x)dx^\mu \wedge dx^\nu$ is the B-field, and dilaton is fixed to be constant. Also in the case of superstring theory this equation still holds true provided all RR background fields vanish [37].

D-branes in group manifold $SU(2)$ (at the semi-classical limit) are determined as wrapping the conjugacy classes of $SU(2)$, which are 2-spheres S^2 plus 2 points-poles, seen as degenerated 2-spheres. Due to the quantization conditions there are $k+1$ D-branes on the level k $SU(2)$ WZW model [27, 37, 2]. To

grasp the dynamics of the branes one should deal with the gauge theory on the stack of N D-branes on S^3 which is quite similar to the flat space case where noncommutative gauge theory emerges [12].

For N branes of type J on top of each other, where J is the representation of $SU(2)_k$ i.e. $J = 0, \frac{1}{2}, 1, \dots, \frac{k}{2}$, the dynamics of the branes is described by the noncommutative action:

$$S_{N,J} = S_{YM} + S_{CS} = \frac{\pi^2}{k^2(2J+1)N} \left(\frac{1}{4} \text{tr}(F_{\mu\nu} F^{\mu\nu}) - \frac{i}{2} \text{tr}(f^{\mu\nu\rho} \text{CS}_{\mu\nu\rho}) \right). \quad (2)$$

Here the curvature form $F_{\mu\nu}(A) = iL_\mu A_\nu - iL_\nu A_\mu + i[A_\mu, A_\nu] + f_{\mu\nu\rho} A^\rho$ and the noncommutative Chern-Simons action reads $\text{CS}_{\mu\nu\rho}(A) = L_\mu A_\nu A_\rho + \frac{1}{3} A_\mu [A_\nu, A_\rho]$. The fields A_μ , $\mu = 1, 2, 3$ are defined on fuzzy 2-sphere S_J^2 and should be considered as $N \times N$ matrix-valued, i.e. $A_\mu = \sum_{j,a} a_{j,a}^\mu Y_a^j$ where Y_a^j are fuzzy spherical harmonics and $a_{j,a}^\mu$ are Chan-Paton matrix-valued coefficients. L_μ are generators of the rotations on fuzzy 2-spheres and they act only on fuzzy spherical harmonics [37]. The noncommutative action S_{YM} was derived from Connes spectral triples of the noncommutative geometry and was aimed to describe Maxwell theory on fuzzy spheres [20].

One can solve the equations of motion derived from the stationery points of (2) and the solutions describing the dynamics of the branes, i.e. the condensation processes on the brane configuration (N, J) which results in another configuration (N', J') . Namely the equation of motion derived from (2) read:

$$L_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0 \quad (3)$$

A class of solutions for (3), in the semi-classical $k \rightarrow \infty$ limit, can be obtained from the $N(2J+1)$ dimensional representations of the algebra $\text{su}(2)$. For $J = 0$ one has N branes of type $J = 0$, i.e. N point-like branes in S^3 at the identity of the group. Given another solution corresponding to the $J_N = \frac{N-1}{2}$ one can show that this corresponds to the brane wrapping the $S_{J_N}^2$ sphere and is obtained as the condensed state of N point-like branes at the identity of $SU(2)$ [37]:

$$(N, J) = (N, 0) \rightarrow (1, \frac{N-1}{2}) = (N', J') \quad (4)$$

Turning to the finite k stringy regime of the $SU(2)$ WZW model one can make use of the techniques of the boundary CFT when applied to the analysis of Kondo effect [37]. It follows that there exists a continuous shift at the level of partition function, between $N\chi_j(q)$ and the interfered sum of characters $\sum_j N_{J_N j}^l \chi_l(q)$ where $N = 2J_N + 1$ (in the vanishing value of the coupling constant) and $N_{J_N j}^l$ are Verlinde fusion rule coefficients. In the case of N point-like branes one can determine the decay product of these by considering open strings ending on the branes. The result on the partition function is

$$Z_{(N,0)}(q) = N^2 \chi_0(q)$$

which is continuously shifted to $N\chi_{J_N}(q)$ and next to $\sum_j N_{J_N J_N}^j \chi_j(q)$. As the result we have the decay process [37]

$$\begin{aligned} Z_{(N,0)}(q) &\rightarrow Z_{(1,J_N)} \\ (N,0) &\rightarrow (1,J_N) \end{aligned} \quad (5)$$

which extends the similar process derived at the semi-classical $k \rightarrow \infty$ limit in the effective gauge theory (4), however the representations $2J_N$ are bounded now, from the above, by k .

Given the above dynamics of branes in the WZW $SU(2)$ model at stringy regime, one can address the question of brane charges in a direct way. This is based on the decay rule (5) in the supersymmetric WZW $SU(2)$ model. In this case we have a shift of the level namely $k \rightarrow k+2$ which measures the units of the NSNS flux through $SU(2) = S^3$. One can see the supersymmetric model as strings moving on $SU(2)$ with $k+2$ units of NSNS flux. From the CFT point of view there exist currents J^a which satisfy $k+2$ level of the Kac-Moody algebra and free fermionic fields ψ^a in the adjoint representation of $su(2)$. However it is possible to redefine the bosonic currents as

$$J^a + \frac{i}{k} f_{bc}^a \psi^b \psi^c$$

which fulfill the current algebra commutation relation at the level k . Here f_{bc}^a are the structure constants of $su(2)$. The fields ψ^a commute with such currents, thus we have the splitting of the supersymmetric WZW $SU(2)$ model at level $k+2$ as WZW $SU(2)$ model at level k times the theory of free fermionic fields.

Thus there are $k+1$ stable branes wrapping the conjugacy classes numbered by $J = 0, \frac{1}{2}, \dots, \frac{k}{2}$. The decaying process (5) says that placing N point-like branes (each charged by the unit 1) at the pole e they can decay to the spherical brane J_N wrapping the conjugacy class. Taking more point-like branes to the stack at e gives the more distant S^2 branes until reaching the opposite pole $-e$ where we have single point-like brane with the opposite charge -1 . Having identify $k+1$ units of the charge with -1 we arrive at the conclusion that the group of charges is \mathbb{Z}_{k+2} . More generally the charges of branes on the background X with non-vanishing $H \in H^3(X, \mathbb{Z})$ are described by the twisted K group $K_H^*(X)$ (see e.g. [13]). In the case of $SU(2)$ we get the group of RR charges as above for $K = k+2$

$$K_H^*(S^3) = \mathbb{Z}_K \quad (6)$$

Based on [10], the following important observation is in order: *certain small exotic \mathbb{R}^4 's generate the group of RR charges of D-branes in the curved background of $S^3 \subset \mathbb{R}^4$.* This observation is based on the integral classes $H \in H^3(S^3, \mathbb{Z})$ from which one can construct the exotic \mathbb{R}_H^4 as corresponding to the codimension-1 foliation of S^3 (determined by the class H). In [10] we showed that twisted K-theory of S^3 by the class $H \in H^3(S^3, \mathbb{Z})$ can be seen as the effect of the exotic smoothness \mathbb{R}_H^4 on the ambient 4-space, when S^3 is understood as the part of the boundary of the Akbulut cork of \mathbb{R}_H^4 .

Thus we arrive at the correspondence:

Theorem 1. *The classification of RR charges of the branes on group manifold background $SU(2)$ at the level k , hence the dynamics of D-branes in S^3 in stringy regime, is correlated with exotic smoothness on \mathbb{R}^4 containing this $S^3 = SU(2)$ as the part of the boundary of the Akbulut cork.*

We can give yet another interpretation of the 4-exoticness which appears on flat \mathbb{R}^4 in this context. Exotic smoothness of \mathbb{R}^4 , \mathbb{R}^4_H , determines the collection of stable D-branes on $SU(2)$ at the level k of the WZW model, where $k = [H] \in H^3(S^3, \mathbb{Z})$. Thus, the stringy, finite k , level of WZW model characterizes exotic 4-smoothness. Recall that in the case of $H = 0$ (e.g. B constant in a flat space, i.e. in $k \rightarrow \infty$ limit) the smooth structure on \mathbb{R}^4 is the standard one [10]. Thus the exotic smoothness on \mathbb{R}^4 translates the 4-curvature to the non-zero H-field on S^3 of finite volume in string units. This is similar to the effect of string field equations relating R and H as in (1), though it holds now between different spaces (\mathbb{R}^4 and S^3).

2.2. $SU(2)$ WZW model in the geometry of the stack of NS5-branes

The group manifold $SU(2) = S^3$ is the only manifold which became relevant so far for the description of small exotic \mathbb{R}^4 . From the other side it is the only one which appears directly as part of a string background (namely one generated by NS5-branes). The reason is given by the connection of 4-exotics and string theory as it can be naturally formulated in the geometry of the stack of NS5-branes. Let us briefly describe this string background [27, 37, 16].

We consider a configuration of k coincident supersymmetric NS5-branes in type II theory. The full fivebrane background is (in string frame)

$$\begin{aligned} ds^2 &= dx^2 + f(r)dy^2 \\ e^{2\phi} &= g_s^2 f(r) \\ f(r) &= 1 + \frac{k\alpha'}{r^2} \\ H_{IJK} &= k\alpha'\epsilon_{IJK} \end{aligned} \tag{7}$$

where x are the $5 + 1$ longitudinal coordinates along NS5-branes referred to by indices μ, ν , etc., y being 4 transverse coordinates referred to by indices $I, J, K \dots$ and $r = |y|$, $1/\alpha' \sim$ string tension. The fields of this background reads as

$$\begin{aligned} e^{2\Phi} &= 1 + \sum_{j=1}^k \frac{l_s^2}{|y - y_j|^2} \\ g_{IJ} &= e^{2\Phi} \delta_{IJ} \\ g_{\mu\nu} &= \eta_{\mu\nu} \\ H_{IJK} &= -\epsilon_{IJKL} \partial^L \Phi \end{aligned} \tag{8}$$

where $y_j, j = 1, \dots, k$ are the positions of the NS5-branes. When the branes coincide at 0, $y_j = 0$, the near horizon solutions $y \rightarrow 0$, are

$$\begin{aligned}
e^{2\Phi} &= \frac{kl_s^2}{|y|^2} \\
g_{IJ} &= e^{2\Phi} \delta_{IJ} \\
g_{\mu\nu} &= \eta_{\mu\nu} \\
H_{IJK} &= -\epsilon_{IJKL} \partial^L \Phi
\end{aligned} \tag{9}$$

In the near-horizon limit $r = |y|^2 \rightarrow 0$, the background factorizes into a radial component and a S^3 and flat 6-dimensional Minkowski spacetime. Strings propagating at this limiting background are described by the exact world-sheet CFT with the target $\mathbb{R}^{5,1} \times \mathbb{R}_\phi \times S_k^3$. Here \mathbb{R}_ϕ is the real line with the parameter ϕ which is a scalar corresponding to the „linear dilaton”

$$\begin{aligned}
\Phi &= -\sqrt{\frac{1}{2k}} \phi \\
\phi &= \sqrt{\frac{k}{2}} \log \frac{r}{kl_s^2}
\end{aligned} \tag{10}$$

The flat Minkowski space $\mathbb{R}^{5,1}$ is longitudinal to the directions of NS5-branes, S_k^3 is $SU(2)_k$ and is a level k WZW supersymmetric CFT (SCFT) on $SU(2)$ as discussed in the previous subsection. This S^3 corresponds to the angular coordinates of the transversal \mathbb{R}^4 . We see that infinite geometrical „throat” $\mathbb{R}_\phi \times S_k^3$, emerges. The metric of the background (in the string frame) thus reads

$$ds^2 = dx_6^2 + d\phi^2 + kl_s d\Omega_3^2, \quad g_s^2(\phi) = e^{-2\phi/\sqrt{k}l_s}.$$

This background is obtained in the near horizon, $\phi \rightarrow -\infty$ ($r \rightarrow 0$), geometry of the stack of $k - 2$ NS5-branes in type II string theory and is in fact a SCFT on the throat. The NS5-branes are placed at $\phi \rightarrow -\infty$ and string theory is strongly coupled there, $g_s \sim \exp(2\Phi)$. In the opposite limit $\phi \rightarrow +\infty$, or $r \rightarrow +\infty$, gives asymptotically flat 10-space and string theory is weakly coupled in that limit. This is essentially the CHS (Callan, Harvey, Strominger [17]) exact string theory background where $SU(2)$ WZW model appears at suitable level k .

Given the CHS limiting geometry of N NS5-branes we have the 4-dimensional tube $\mathbb{R}_\phi \times S^3$. The volume of S^3 in string units is finite and correlated with the number of NS5-branes by $N = k - 2$ [16]. We take an exotic \mathbb{R}_H^4 for $[H] = k \in H^3(S^3, \mathbb{Z})$. This can be achieved more directly by considering the Akbulut cork A_H with the boundary, $\partial A_H = \Sigma_H$, the homology 3-sphere. As was shown in [10] Σ_H contains S^3 such that the codimension-1 foliations of it generates the foliations of Σ_H . The foliations in turn are generated by Casson handles attached to A . Thus the attached Akbulut cork and Casson handle(s) determine the small exotic smoothness of \mathbb{R}_H^4 [29, 10]. Moreover, the cobordism classes of codimension-1 foliations of S^3 are classified by the Godbillon-Vey invariants which are elements of $H^3(S^3, \mathbb{R})$. In our case we deal with integral 3-rd cohomologies $[H] \in H^3(S^3, \mathbb{Z})$. Thus, a way of embedding the Akbulut cork, for some class of exotic \mathbb{R}^4 's, in the ambient \mathbb{R}^4 is determined by the integral classes $k \in H^3(S^3, \mathbb{Z})$. Taking the above S^3 from the boundary of the Akbulut

cork, as $S^3 = SU(2)$ in the string background of N NS5-branes we arrive at the following result:

Theorem 2. *In the geometry of the stack of NS5-branes in type II superstring theories, adding or subtracting a NS5-brane is correlated with the change of smoothing on transversal \mathbb{R}^4 .*

Now the tube $\mathbb{R}_\phi \times S_k^3$ of the limiting geometry can be embedded in the ambient standard \mathbb{R}^4 . Taking this S_k^3 as lying in the boundary of the Akbulut cork for some exotic smooth \mathbb{R}_H^4 , the embedding of the tube in this exotic 4-space is determined by the embedding of the Akbulut cork. But this embedding is determined by Casson handles attached to the cork and corresponds to the integral class $[H] = k \in H^3(S^3, \mathbb{Z})$. Thus the background $\mathbb{R}^{5,1} \times \mathbb{R}_\phi \times SU(2)_k$ is geometrically realized as $\mathbb{R}^{5,1} \times \mathbb{R}_H^4$. We propose here a general heuristic rule:

R1. *D-branes probing exotic 4-dimensional Euclidean space, \mathbb{R}_H^4 , times 6-dim. Minkowski spacetime $M^{5,1}$, are described equivalently by the D-branes of type II string theory probing the transversal 4-space, \mathbb{R}^4 , to k NS5-branes in the background of these 5-branes. Here $[H] = k \in H^3(S^3, \mathbb{Z})$. Since $M^{5,1}$ appears in both sides of the correspondence we say that D-branes explore exotic Euclidean \mathbb{R}_H^4 .*

Rule R1 is based on the assumption that various nonstandard smoothings of \mathbb{R}^4 can be grasped by the effects of $H^3(S^3, \mathbb{Z})$. This follows from the correlation of the classes and 4-exotics as proved in [10]. Following this rule we can consider many examples of D-branes in the above background (see e.g. [4, 25, 32, 35, 21]), as referring to 4-exoticness.

Furthermore type II string theory on $\mathbb{R}^{5,1} \times \mathbb{R}_\phi \times SU(2)_k$ is given by the SCFT on the infinite „throat” of the background, i.e. $\mathbb{R}_\phi \times S^3$. Then this theory was proposed to be approachable via holography by using duality. The holographically dual theory appears to be so called 6-dimensional *little string theory* (LST) [25, 1]. This is a very interesting situation for us since LST was analyzed as having possible experimental signatures at the TeV scales after the compactification on torus [4]. By the rule above this refers to 4-exotics as well. We do not deal here with the details and refer the interesting reader to a separate paper devoted to (flux) compactification in string theory and exotic 4-smoothness. But we will present some general remarks here.

LST's are non-local theories without gravity and can be described in the limit $g_s \rightarrow 0$ in the theory on k NS5-branes. In that limit the bulk degrees of freedom decouple, hence gravity does. This 6-dim. LST without gravity is holographically dual to the type II string theory formulated on the background $\mathbb{R}^{5,1} \times \mathbb{R}_\phi \times SU(2)_k$ [4]. From the rule R1 it follows that LST is referred to exotic \mathbb{R}_H^4 and calculations in LST should lead to invariants of the 4-exotics. The perturbative calculations, however, are hardly performed in LST since the string coupling g_s diverges in the dual string background along the tube, and LST is sensitive on that. One usually regulates the geometry via chopping the tube. But the decomposition of the SCFT $SU(2)_k$ on $S_Y^1 \times SU(2)_k/U(1)$ can be performed. Here $SU(2)_k/U(1)$ is a minimal $N = 2$ model at the level k and

S_Y^1 is the Cartan subalgebra of $SU(2)$ with the parameter Y . The dependence on k is crucial at this reformulation since this refers to 4-exotics by theorem 2 and the rule R1. Thus we have the SCFT $\mathbb{R}_\phi \times S_Y^1 \times \frac{SU(2)_k}{U(1)}$ instead of the tube $\mathbb{R}_\phi \times SU(2)_k$. The chopping of the strong coupling region is now performed by taking the SCFT $\frac{SL(2)_k}{U(1)}$ instead of $\mathbb{R}_\phi \times S_Y^1$ which means replacing the background $\mathbb{R}^{5,1} \times \mathbb{R}_\phi \times SU(2)_k$ by $\mathbb{R}^{5,1} \times \frac{SL(2)_k}{U(1)} \times \frac{SU(2)_k}{U(1)}$. This means, on the level of k NS5-branes, the separation of these 5-branes along the transverse circle of radius L . Now the double-scale limit of LST is the one when taking both g_s and L to zero while $\frac{L}{g_s}$ remains constant.

Following [25] we can take systems of D4, D6-branes between separated NS5-branes. The various expressions like correlation functions can be now calculated perturbatively in the holographically dual 6-dimensional LST theory. Besides, suitable compactifications may refer to the spectra with the TeV scale of the standard model of particles. The dependence on k of some of these expressions can be seen as the signature of the existence of exotic structure in the 4-space transversal to the branes.

Exoticness of the 4-space transversal to the worldvolume of NS5-branes, is reflected in specific perturbative spectra of D-branes when calculated in dual 6-dimensional LST. When compactifying this LST on 2 directions longitudinal to the 5-brane one gets spectra which could be sensitive on transversal exoticness of \mathbb{R}^4 . From the point of view of physics, the calculations refer to the TeV scale [1].

The important observation can be made: *Some LST calculations refer not only to holographically dual string theory but also to exotic smoothness on \mathbb{R}^4 . This is the indication that one can try, at least in some cases, to replace higher dimensional string theory effects by 4-dimensional phenomena.*

This is in fact the reformulation of the rule R1. The NS5-branes backgrounds show that string theory computations „feel” the 4-exoticness.

3. Quantum D-branes and 4-exotica

In this section we want to show that D-branes of string theory, as in the previous sections, are related with exotic smooth \mathbb{R}^4 's also beyond the semi-classical limit, i.e. in the quantum regime of the theory where one should deal rather with *quantum branes*. What *quantum branes* mean in general is still an open and hard problem. One appealing proposition, relevant for this paper, is to consider branes in noncommutative spacetimes rather than on commutative manifolds or orbifolds. This leads to abstract D-branes in general noncommutative separable C^* algebras as counterparts for quantum D-branes. In the next section we will present a definition using wild embeddings.

3.1. D-branes on spaces: K-homology and KK theory

The description of systems of stable Dp-branes of IIA,B string theories via K-theory of topological spaces can be extended toward the branes in noncommutative C^* algebras. A direct string representation of the algebraic and K-theoretic

ideas is best seen in K-matrix string theory where, in particular, tachyons are elements of the spectral triples representing the noncommutative geometry of the world-volumes of the configurations of branes [5]. The elements of the formulation of type II strings as K matrix theory is presented in the Appendix A.

First let us consider the case of vanishing H -field on X . The charges of D-branes are classified by suitable K theory groups, i.e. $K^0(X)$ in IIB and $K^1(X)$ in IIA string theories, where X is the background manifold. On the other hand, world-volumes of Dp-branes correspond to the cycles of K homology groups, $K_1(X)$, $K_0(X)$, which are dual to the K theory groups. Let us see how K -cycles correspond to the configurations of D-branes.

A K - cycle on X is a triple (M, E, ϕ) where M is a compact Spin^c manifold without boundary, E is a complex vector bundle on M and $\phi : M \rightarrow X$ is a continuous map. The topological K -homology $K_\star(X)$ is the set of equivalence classes of the triples (M, E, ϕ) respecting the following conditions:

- (i) $(M_1, E_1, \phi_1) \sim (M_2, E_2, \phi_2)$ when there exists a triple (bordism of the triples) (M, E, ϕ) such that $(\partial M, E|_{\partial M}, \phi|_{\partial M})$ is isomorphic to the disjoint union $(M_1, E_1, \phi_1) \cup (-M_2, E_2, \phi_2)$ where $-M_2$ is the reversed Spin^c structure of M_2 and M is a compact Spin^c manifold with boundary.
- (ii) $(M, E_1 \oplus E_2, \phi) \sim (M, E_1, \phi) \cup (M, E_2, \phi)$,
- (iii) Vector bundle modification $(M, E, \phi) \sim (\widehat{M}, \widehat{H} \otimes \rho^*(E), \phi \circ \rho)$. \widehat{M} is even dimensional sphere bundle on M , $\rho : \widehat{M} \rightarrow M$ projection, \widehat{H} is a vector bundle on \widehat{M} which gives the generator of $K(S_q^{2p}) = \mathbb{Z}$ on every S_q^{2p} over each $q \in M$ [38].

The topological K-homology as above has an abelian group structure with disjoint union of cycles as sum. The triples (M, E, ϕ) with M being even dimensional determines $K_0(X)$. Similarly, $K_1(X)$ corresponds to odd dimensions. Thus $K_\star(X)$ decomposes into a direct sum of abelian groups:

$$K_\star(X) = K_0(X) \oplus K_1(X).$$

Now the interpretation of cycles (M, E, ϕ) as D-branes [31] is the following: M is the world-volume of brane, E the Chan-Paton bundle on it and ϕ gives the embedding of the brane into spacetime X . Moreover, M has to wrap Spin^c manifold [28] and $K_0(X)$ classifies stable D-branes configurations in IIB, and $K_1(X)$ in IIA, string theories. The equivalences of K-cycles as formulated in the conditions (i)-(iii) correspond to natural relations for D-branes [5, 14].

The topological K-homology theory above can be obtained analytically (analytic K-homology theory) as a special commutative case of the following construction on general C^\star algebras [5].

A Fredholm module over a C^\star algebra \mathcal{A} is a triple (\mathcal{H}, ϕ, F) such that

1. \mathcal{H} is a separable Hilbert space,

2. ϕ is a $*$ homomorphism between C^* algebras \mathcal{A} and $\mathbf{B}(\mathcal{H})$ of bounded linear operators on \mathcal{H} ,
3. F is self-adjoint operator in $\mathbf{B}(\mathcal{H})$ satisfying

$$F^2 - id \in \mathbf{K}(\mathcal{H}), \quad [F, \phi(a)] \in \mathbf{K}(\mathcal{H}) \text{ for every } a \in \mathcal{A}$$

where $\mathbf{K}(\mathcal{H})$ are compact operators on \mathcal{H} . Now let us see how a Fredholm module (\mathcal{H}, ϕ, F) describes certain configuration of IIA K matrix string theory directly related to D branes. To this end we consider the operators of the K-matrix theory Φ^0, \dots, Φ^9 (infinite matrices) acting on the Hilbert space \mathcal{H} as generating the C^* algebra \mathcal{A}_M (see the Appendix AppendixA and [5]). In the case of commuting Φ^μ , hence commutative \mathcal{A}_M , we have the following correspondence (explaining the index M in \mathcal{A}_M):

- Every commutative C^* algebra is isomorphic to the algebra of continuous complex functions vanishing at infinity $C(M)$ on some locally compact Hausdorff space M (Gelfand-Najmark theorem). A point $x \in M$ is determined by a character of \mathcal{A}_M which is a $*$ homomorphism $\phi_x : \mathcal{A}_M \rightarrow \mathbb{C}$.
- M serves as a common spectrum for Φ^0, \dots, Φ^9 and the choice of a point from M as the eigenvalue of Φ^μ fixes the position of the non BPS instanton along x^μ .
- In this way M is covered by the positions of infinite many non BPS instantons and serves as the world-volume of some higher dimensional D brane [5].

Now let us explain the role of the tachyon T . T is a self-adjoint unbounded operator acting on the Chan-Paton Hilbert space \mathcal{H} . \mathcal{A}_M is a C^* unital algebra generated by Φ^0, \dots, Φ^9 which can be now noncommutative. The corresponding geometry of the world-volume M would be noncommutative and given by some spectral triple. The spectral triple is in fact $(\mathcal{H}, \mathcal{A}, T)$ which means that the following conditions are satisfied [5]:

$$T - \lambda \in \mathbf{K}(\mathcal{H}) \text{ for every } \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad [a, T] \in \mathbf{B}(\mathcal{H}) \text{ for every } a \in \mathcal{A}_M$$

These conditions indeed hold true in our case of K matrix string theory for a tachyon field T , Chan-Paton Hilbert space \mathcal{H} and C^* algebra \mathcal{A}_M generated by Φ^μ (see Appendix AppendixB). The extension of spacetime manifold toward noncommutative algebra and noncommutative world-volumes of branes, represented by spectral triples, is thus given by [5]:

1. Fixing the spacetime C^* algebra \mathcal{A} ;
2. A $*$ homomorphism $\phi : \mathcal{A} \rightarrow \mathbf{B}(\mathcal{H})$ generates embedding of the D-brane world-volume M and its noncommutative algebra \mathcal{A}_M as $\mathcal{A}_M := \phi(\mathcal{A})$;
3. D-branes embedded in a spacetime \mathcal{A} are represented by the spectral triple $(\mathcal{H}, \mathcal{A}_M, T)$;

4. Equivalently, D-brane in A is given by unbounded Fredholm module (\mathcal{H}, ϕ, T) .

In particular the classification of stable D-branes in \mathcal{A} is the classification of Fredholm modules (\mathcal{H}, ϕ, T) given by analytical K-homology. Given the isomorphisms of the topological and analytical K homology groups, we have the classification of stable D-branes in terms of K-cycles, as we discussed at the beginning of this section. In terms of K matrix string theory we can say that stable configurations of D-instantons determine the stable higher dimensional D-branes which are K-homologically classified as above.

Now let us turn to a more general situation than K-string theory of D-instantons, i.e. backgrounds given by non-BPS Dp-branes or non-BPS Dp- $\overline{\text{Dp}}$ -branes in type II string theory. The stable configurations of Dq-branes are then classified by generalized K-theory namely Kasparov KK-theory. As in the above case of D-branes in a C^* algebra \mathcal{A} corresponding to Fredholm modules, one defines an odd Kasparov module $(\mathcal{H}_{\mathcal{B}}, \phi, T)$, where $\mathcal{H}_{\mathcal{B}}$ is a countable Hilbert module over C^* algebra \mathcal{B} , as

- a \star -homomorphism from \mathcal{A} to the C^* algebra of bounded linear operators on $\mathcal{H}_{\mathcal{B}}$, $\phi : \mathcal{A} \rightarrow \mathbf{B}(\mathcal{H}_{\mathcal{B}})$;
- a self-adjoint operator T from $\mathbf{B}(\mathcal{H}_{\mathcal{B}})$ satisfying:

$$T^2 - 1 \in \mathbf{K}(\mathcal{H}_{\mathcal{B}}) \text{ and } [T, \phi(a)] \in \mathbf{K}(\mathcal{H}_{\mathcal{B}}) \text{ for every } a \in \mathcal{A},$$

where $\mathbf{K}(\mathcal{H}_{\mathcal{B}})$ is $\mathcal{B} \otimes \mathbf{K}$. $(\mathcal{H}_{\mathcal{B}}, \phi, T)$ is in fact a family of Fredholm modules on the algebra \mathcal{B} . When \mathcal{B} is \mathbb{C} we have an ordinary Fredholm module as before. The homotopy equivalence classes of odd Kasparov modules $(\mathcal{H}_{\mathcal{B}}, \phi, T)$ determine elements of $KK^1(\mathcal{A}, \mathcal{B})$. Also one defines an even Kasparov classes $KK^0(\mathcal{A}, \mathcal{B}) = KK(\mathcal{A}, \mathcal{B})$ as homotopy equivalence classes of the triples $(\mathcal{H}_{\mathcal{B}}^{(0)} \oplus \mathcal{H}_{\mathcal{B}}^{(1)}, \phi^{(0)} \oplus \phi^{(1)}, \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix})$. A natural \mathbb{Z}_2 grading appears due to the involution $\mathcal{H}_{\mathcal{B}}^{(0)} \oplus \mathcal{H}_{\mathcal{B}}^{(1)} \rightarrow \mathcal{H}_{\mathcal{B}}^{(0)} \oplus -\mathcal{H}_{\mathcal{B}}^{(1)}$.

Now the classification pattern for branes in spaces emerges. There are non-BPS unstable Dp-branes wrapping the $p+1$ -dimensional world-volume B . Then stable Dq-branes configurations embedded in a space A transverse to B correspond to (are classified by) the classes of $KK^1(A, B)$. Similarly, given non-BPS unstable Dp- $\overline{\text{Dp}}$ -branes system, then stable Dq-branes embedded in A transverse to B ($p+1$ -dimensional world-volumes) are classified by elements of $KK^0(A, B)$. The case of even $KK^0(A, B)$ contains the \mathbb{Z}_2 grading as corresponding to the Chan-Paton indices of Dp and $\overline{\text{Dp}}$ -branes.

3.2. D-branes on separable C^* algebras and KK theory

The classification of D-branes in a spacetime manifold given by KK theory as sketched in the previous subsection, can be extended over noncommutative spacetimes and noncommutative D-branes both represented by separable C^*

algebras. Let us first recapitulate the „classic” case of spaces allowing the extension over C^* algebras [39].

In the case of type II superstring theory, let X be a compact part of spacetime manifold, i.e. X is a compact spin^c manifold again with no background H -flux. As we saw, a flat D-brane in X is a Baum-Douglas K-cycle (W, E, f) . Here $f : W \hookrightarrow X$ is the embedding of the closed spin^c submanifold W of X and $E \rightarrow W$ is a complex vector bundle with connection (Chan-Paton gauge bundle). As follows from Baum-Douglas construction, E determines the stable class in the K-theory group $K^0(W)$ and all K-cycles form an additive category under disjoint union. Now, the set of all K-cycles classes up to a kind of gauge equivalence as in Baum-Douglas construction, gives the K-homology of X . This K-homology is also the set of stable homotopy classes of Fredholm modules which are taken over the commutative C^* algebra $C(X)$ of continuous functions on X . This defines the correspondence (isomorphism) where a K-cycle (W, E, f) corresponds to unbounded Fredholm module $(\mathcal{H}, \rho, D_E^W)$. Here \mathcal{H} is the separable Hilbert space of square integrable spinors on W taking values in the bundle E , i.e. $L^2(W, S \otimes E)$, $\rho : C(X) \rightarrow \mathbf{B}(\mathcal{H})$ is the representation of the C^* algebra $C(X)$ in \mathcal{H} such that $C(X) \ni g \rightarrow a_{g \circ f} \in \mathbf{B}(\mathcal{H})$ where $a_{g \circ f}$ is the operator of point-wise multiplication of functions in $L^2(W, S \otimes E)$ by the function on W , $g \circ f$, and $f : W \hookrightarrow X$. D_E^W is the Dirac operator twisted by E corresponding to the spin^c structure on W . Given the K-theory class of the Chan-Paton bundle E , i.e. $[E] \in K^0(W)$, then the dual K-homology class of a D-brane, $[W, E, f]$ uniquely determines $[E]$. In that way D-branes determine K-homology classes on X which are dual to K-theory classes from $K^r(X)$ where r is the transversal dimension for the brane world-volume W . This K-theory class is derived from the image of $[E] \in K^0(W)$ by the Gysin K-theoretic map $f_!$. As we discussed already, the odd and even classes of K-homology $K_*(X)$ correspond to the parity of the dimension of W . The K-cycle (W, E, f) corresponds to a Dp-brane and its gauge equivalence is given by Baum-Douglas construction using the conditions (i)-(iii) in Sec. 3.1. Thus we have [14]:

Fact 1: *There is a one-to-one correspondence between flat D-branes in X , modulo Baum-Douglas equivalence, and stable homotopy classes of Fredholm modules over the algebra $C(X)$.*

In the presence of a non-zero B -field on X , which is a $U(1)$ -gerbe with connection represented by the characteristic class in $H^3(X, \mathbb{Z})$ [14, 10], one can define twisted D-brane on X as [14]:

Definition 1. *A twisted D-brane in a B-field (X, H) is a triple (W, E, ϕ) , where $\phi : W \hookrightarrow X$ is a closed, embedded oriented submanifold with $\phi^*H = W_3(W)$, and E is the Chan-Paton bundle on W , i.e. $E \in K^0(W)$, and $W_3(W)$ is the 3-rd integer Stiefel-Whitney class of the normal bundle of W , $W_3(W) \in H^3(W, \mathbb{Z})$.*

The condition in the definition is in fact required by the cancellation of the Freed-Witten anomaly, where $H \in H^3(X, \mathbb{Z})$ is the NS-NS H -flux. Since $W_3(W)$ is the obstruction to the spin^c structure on W , in the case of $W_3(W) = 0$ one has flat D-branes in X . Thus equivalence classes of twisted D-branes

on X are represented by twisted topological K-homology $K_*(X, H)$ which is dual to the twisted K-theory $K^*(X, H)$. As was argued in [8], in case of S^3 , one has some exotic \mathbb{R}^4 's which can be twisted by H leading to the K-theory $K^*(S^3, H)$. We can represent the $U(1)$ gerbes with connection on S^3 , by the bundles \mathcal{E}_H of algebras over S^3 , such that the sections of the bundle \mathcal{E}_H define the noncommutative, twisted algebra $C_0(X, \mathcal{E}_H)$ and the Dixmier-Douady class of \mathcal{E}_H , $\delta_H(\mathcal{E}_H)$, is $H \in H^3(S^3, \mathbb{Z})$ [9, 11, 38]. The important relation is the following ([14], Proposition 1.15):

Fact 2: There is a one-to-one correspondence between twisted D-branes in (X, H) and stable homotopy classes of Fredholm modules over the algebra $C_0(X, \mathcal{E}_H)$.

Since the algebra $C_0(X, \mathcal{E}_H)$ certainly determines its stable homotopy classes of the Fredholm modules on it, then in the case $X = S^3$ one has the following observation:

A. Let the exotic smooth \mathbb{R}^4 's are determined by the integral third classes $H \in H^3(S^3, \mathbb{Z})$. Then, these exotic smooth \mathbb{R}^4 's correspond one-to-one to the set of twisted D-branes in (S^3, H) .

In principle, given the complete collection of twisted D-branes in (S^3, H) , which take values in $K_*(S^3, H)$, one can determine the corresponding exotic \mathbb{R}^4 . This is simply the exotic \mathbb{R}^4_H corresponding to the class $[H] \in H^3(S^3)$ and H makes the twist in the K-homology as dual to the twisted K-theory $K^*(S^3, H)$ [9, 8, 38]. In this paper we collect further evidences that this is also the case more generally, and the relation D-branes - 4-exotics is closer.

Remembering that $S^3 \subset \mathbb{R}^4$ as part of the Akbulut cork of the exotic structure, our previous observation can be restated as:

B. The change of the exotic smoothness of \mathbb{R}^4 , $\mathbb{R}^4_{H_1} \rightarrow \mathbb{R}^4_{H_2}$, $H_1, H_2 \in H^3(S^3, \mathbb{Z})$, $H_1 \neq H_2$, corresponds to the change of the curved backgrounds $(S^3, H_1) \rightarrow (S^3, H_2)$ hence the sets of stable D-branes.

This motivates the formulation:

C. Some small exotic smoothness on \mathbb{R}^4 , $\mathbb{R}^4_{H_1}$, can be destabilize (or stabilize) D-branes in (S^3, H_2) , where $S^3 \subset \mathbb{R}^4$ lies at the boundary of the Akbulut cork of $\mathbb{R}^4_{H_1}$. We say that D-branes in (S^3, H_2) are 4-exotic-sensitive.

Turning to the generalization of spaces to noncommutative C^* algebras, there were developed recently impressive counterparts of many topological, geometrical and analytical results, like Poincaré duality, characteristic classes and the Riemann-Roch theorem. Also the generalized formula for charges of quantum D-branes in a noncommutative separable C^* algebras was worked out [15, 14]. Thus the suitable framework for considering the quantum regime of D-branes emerged. In next subsection we will try to find a relation to 4-exotics also in this quantum regime of D-branes.

Following [5, 15, 14, 39] one can take as an initial substitute for the category of quantum D-branes, the category of separable C^* algebras and morphisms being elements of KK theory groups. This means that for a pair $(\mathcal{A}, \mathcal{B})$ of separable C^* algebras the morphisms $h : \mathcal{A} \rightarrow \mathcal{B}$ is lifted to the element of the group $KK(\mathcal{A}, \mathcal{B})$. Thus we can consider a generalized D-branes in a separable C^* algebra \mathcal{A} as corresponding to the lift $h! : \mathcal{A} \rightarrow \mathcal{B}$ where \mathcal{B} represents a quantum D-brane.

More precisely following [15], let us consider a subcategory \mathcal{C} of the category of C^* separable algebras and their morphisms, which consists of strongly K-oriented morphisms. This means that there exists a contravariant functor $! : \mathcal{C} \rightarrow KK$ such that $\mathcal{C} \ni f : \mathcal{A} \rightarrow \mathcal{B}$ is mapped to $f! \in KK_d(\mathcal{B}, \mathcal{A})$, here KK is the category of separable C^* algebras with KK classes as morphisms. Strongly K-oriented morphisms and the functor $!$ are subjects to the following conditions:

1. Identity morphism $id_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}$ is strongly K-oriented (SKKO) for every separable C^* algebra \mathcal{A} and $(id_{\mathcal{A}})! = 1_{\mathcal{A}}$. Also, the 0-morphism $0_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}$ is SKKO and $(0_{\mathcal{A}})! = 0 \in KK(0, \mathcal{A})$.
2. If $f : \mathcal{A} \rightarrow \mathcal{B}$ is SKKO then $f^{\circ} : \mathcal{A}^{\circ} \rightarrow \mathcal{B}^{\circ}$ is either, and $(f!)^{\circ} = (f^{\circ})!$. \mathcal{A}° is the opposite C^* algebra to \mathcal{A} , i.e. one which has the same underlying vector space but reversed product.
3. Any morphism $f : \mathcal{A} \rightarrow \mathcal{B}$ is SKKO, provided \mathcal{A} and \mathcal{B} are strong Poincaré dual (PD) algebras. Then $f!$ is determined as:

$$f! = (-1)^{d_{\mathcal{A}}} \Delta_{\mathcal{A}}^{\vee} \otimes_{\mathcal{A}^{\circ}} [f^0] \otimes_{\mathcal{B}^{\circ}} \Delta_{\mathcal{B}} \quad (11)$$

here $[f]$ is the class of $f : \mathcal{A} \rightarrow \mathcal{B}$ in $KK(\mathcal{A}, \mathcal{B})$. $\Delta_{\mathcal{A}}$ is the fundamental class in $KK_{d_{\mathcal{A}}}(\mathcal{A} \otimes \mathcal{A}^{\circ}, \mathbb{C}) = K^{d_{\mathcal{A}}}(\mathcal{A} \otimes \mathcal{A}^{\circ})$, $\Delta_{\mathcal{A}}^{\vee}$ its dual class in $KK_{-d_{\mathcal{A}}}(\mathbb{C}, \mathcal{A} \otimes \mathcal{A}^{\circ}) = K_{-d_{\mathcal{A}}}(\mathcal{A} \otimes \mathcal{A}^{\circ})$ which exist by strong PD [15].

K-orientability was introduced, in its original form, by A. Connes in order to define the analogue of spin^c structure for noncommutative C^* algebras (see also [22] and next subsections). Presented here formulation of K-orientability and strong PD C^* algebras are crucial ingredients of noncommutative versions of Riemann-Roch theorem, Poincaré-like dualities, Gysin K-theory map and allows to formulate a very general formula for noncommutative D-brane charges [14, 15, 39]. Let us notice that if both \mathcal{A} and \mathcal{B} are PD algebras then any morphism $f : \mathcal{A} \rightarrow \mathcal{B}$ is K-oriented and the K-orientation for f is given in (11).

In the particular case of the proper smooth embedding $f : M \rightarrow X$ of codimension d , where M, X are smooth compact manifolds, let the normal bundle τ over W , of TW with respect to $f^*(TX)$, be spin^c . When also X is spin^c then the spin^c condition on τ when H -flux is absent in type II string theory formulated on X , is the Freed-Witten anomaly cancellation condition [15]. In this case any D-brane in X , given by the triple (W, E, f) , determines the KK-theory element $f! \in KK(C(W), C(X))$. The construction of K-orientation $f : M \rightarrow X$, between smooth compact manifolds, can be extended to smooth proper maps which are not necessary embeddings. Thus the general condition for K-orientability gives the correct analogue for stable D-branes in C^* algebras.

Definition 2. A generalized stable quantum D-brane on a separable C^* algebra \mathcal{A} , represented by a separable C^* algebra \mathcal{B} , is given by the strongly K-oriented homomorphism of C^* algebras, $h_{\mathcal{B}} : \mathcal{A} \rightarrow \mathcal{B}$. The K-orientation means that there is the lift $(h_{\mathcal{B}})! \in KK(\mathcal{B}, \mathcal{A})$ where $!$ fulfills the functoriality condition as in (11).

This kind of an approach to quantum D-branes is in fact a conjectural framework which exceeds both the dynamical Seiberg-Witten limit of superstring theory (where noncommutative brane world-volumes emerges) and geometrical understanding of branes, and places itself rather in a deep quantum regime of the theory [39].

3.3. Exotic \mathbb{R}^4 and stable D-branes configurations on foliated manifolds

Now we want to approach the problem of description of stable states of D-branes in a more general geometry than spaces, namely the geometry of foliated manifolds. The case of our interest is a codimension-1 foliation of S^3 . This is a noncommutative geometry. In general, to every foliation (V, F) one can associate its noncommutative C^* algebra $C^*(V, F)$, on the other hand a foliation determines its holonomy groupoid G and the topological classifying space BG . Both cases, topological K-homology of G and C^* algebraic K-theory, are in fact dual. Analogously to our previous discussion of branes as K-cycles on X , let us start with K-homology of G and define D-branes as K-cycles in G :

A K - cycle on a foliated geometry $X = (V, F)$ is a triple (M, E, ϕ) where M is a compact manifold without boundary, E is a complex vector bundle on M and $\phi : M \rightarrow BG$ is a smooth K-oriented map. Due to the K-orientability in the presence of canonical G -bundle τ on BG , the condition of Spin^c structure on M is lifted to the Spin^c structure on $TM \oplus \phi^* \tau$ [22].

The topological K -homology $K_{*,\tau}(X) = K_{*,\tau}(BG)$ of the foliation (V, F) is the set of equivalence classes of the above triples, where the equivalence respects the following conditions:

- (i) $(M_1, E_1, \phi_1) \sim (M_2, E_2, \phi_2)$ when there exists a triple (bordism of the triples) (M, E, ϕ) such that $(\partial M, E|_{\partial M}, \phi|_{\partial M})$ is isomorphic to the disjoint union $(M_1, E_1, \phi_1) \cup (-M_2, E_2, \phi_2)$ where $-M_2$ is the reversed Spin^c structure of $TM_2 \oplus \phi_2^* \tau$ and M is a compact manifold with boundary.
- (ii) $(M, E_1 \oplus E_2, \phi) \sim (M, E_1, \phi) \cup (M, E_2, \phi)$,
- (iii) Vector bundle modification $(M, E, \phi) \sim (\widehat{M}, \widehat{H} \otimes \rho^*(E), \phi \circ \rho)$ similarly as in the case of manifolds.

As in the case of spaces (manifolds) and the corresponding K-homology groups representing stable D-branes of type II superstring theory (see Sec. 3.1), also here, in the case of the geometry of foliated manifolds we generalize stable D-branes as being represented by the above triples.

Theorem 3. *The class of generalized stable D-branes on the C^* algebra $C^*(S^3, F_1)$ (of the codimension 1 foliation of S^3) which correspond to the K-homology classes $K_{*,\tau}(S^3/F)$, determines an invariant of exotic smooth \mathbb{R}^4 . Such an exotic \mathbb{R}^4 contains this foliated S^3 as a generalized (noncommutative) smooth subset [9].*

The result follows from the fact that $K_{*,\tau}(S^3/F)$ is isomorphic to $K_{*,\tau}(BG)$ [22] and this determines a class of stable D-branes in (S^3, F) . The foliations (S^3, F) correspond to different smoothings on \mathbb{R}^4 [10]. \square

Let us note that this approach allows for considering a kind of string theory and branes also beyond the integral levels of $SU(2)$ WZW model given by $[H] \in H^3(S^3, \mathbb{Z})$. The relation with exotic smooth \mathbb{R}^4 's extends over this as well.

3.4. Net of exotic \mathbb{R}^4 's and quantum D-branes in $C^*(S^3, F)$

The extension of string theory and D-branes over general noncommutative separable C^* algebras where also D-branes are represented by noncommutative separable C^* algebras, can be considered as an approach to quantum D-branes. A category of D-branes in a quantum regime, is the category of separable C^* algebras and morphisms which are elements of KK theory groups. For a pair $(\mathcal{A}, \mathcal{B})$ of separable C^* algebras the morphisms $h : \mathcal{A} \rightarrow \mathcal{B}$ belong to $KK(\mathcal{A}, \mathcal{B})$. Abstract quantum D-branes in a separable C^* algebra \mathcal{A} correspond to $\phi : \mathcal{A} \rightarrow \mathcal{B}$ where \mathcal{B} is the algebra representing a quantum D-brane and ϕ is a strongly K-oriented map. For such branes a general formula for RR charges in noncommutative setting was worked out [15, 14].

D-branes considered in the previous subsection, correspond to the lifted KK-theory classes, i.e. $f! \in KK(M, V/F)$ where D-brane corresponds to the triple (M, E, f) and $f : M \hookrightarrow G = V/F$ is K-oriented map (see [22]). More generally (still following [22]), given a K-oriented map $f : X \rightarrow Y$, one can define (under certain conditions) a push forward map $f!$ in K-theory. The very important property of the analytical group $K(V/F)$ of the foliation (V, F) is its „wrong way” (Gysin) functoriality which to each K-oriented map $f : V_1/F_1 \rightarrow V_2/F_2$ of leaf spaces associates an element $f!$ of the Kasparov group $KK(C^*(V_1; F_1); C^*(V_2; F_2))$.

Now given a small exotic \mathbb{R}^4 , say e_1 , embedded in some small exotic \mathbb{R}^4 , e , both are represented by the C^* algebras of the codimension-1 foliations of S^3 , $C^*(V_1; F_1)$ and $C^*(V; F)$ respectively. The embedding $i : e_1 \hookrightarrow e$ determines the corresponding K-oriented map of the leaf spaces $f_i : S^3/F_1 \rightarrow S^3/F$ and the KK-theory lift $f_i! : KK(C^*(V_1; F_1); C^*(V; F))$. According to Def. 2 from Sec. 3.2, we see that

Theorem 4. *Let e be an exotic \mathbb{R}^4 corresponding to the codimension-1 foliation of S^3 which gives rise to the C^* algebra \mathcal{A}_e . The exotic smooth \mathbb{R}^4 embedded in e determines a generalized quantum D-brane in \mathcal{A}_e .*

Given exotic \mathbb{R}^4 's, $\{e_a, a \in I\}$, all embedded in e , one has the family of C^* algebras, $\{\mathcal{A}_a, a \in I\}$, of the codimension-1 foliations of S^3 , $a \in I$. Now the embeddings $e_a \rightarrow e$ determine the corresponding K-oriented maps of the leaf spaces as before, and the \star -homomorphisms of algebras $\phi_a : \mathcal{A}_e \rightarrow \mathcal{A}_a$. The corresponding classes in KK theory $KK(\mathcal{A}_e, \mathcal{A}_a)$, represent quantum D-branes in \mathcal{A}_e . \square

However, the correspondence in the theorem is many-to-one and an exotic smooth \mathbb{R}^4 embedded in e can be represented (non-uniquely) by stable D-brane

in \mathcal{A}_e , and not all abstract D-branes in the algebra \mathcal{A}_e are represented by some exotic $e' \subset e$. Still one can consider D-branes represented by exotic e_a in e as carrying 4-dimensional, hence potentially physical, information. This is a kind of special „superselection” rule in superstring theory and will be discussed separately.

3.5. RR charges of D6-Branes in the presence of B-field

Now let us comment on some indication how 4-dimensional structure can refer directly to dynamics of higher dimensional branes in flat spacetime. This higher dimensional brane is the important D6-brane which is usually involved in building various „realistic” 4-dimensional models derived from the brane configurations. We will analyze this case separately along with compactifications in string theory.

Let us consider the D6-brane of IIA string theory in flat 10 dimensional spacetime and assume that B-field vanishes. The world-volumes of flat Dp-branes are classified by $K_1(\mathbb{R}^{p+1})$ where this K-homology group is understood as $K^1(C_0(\mathbb{R}^{p+1}))$ i.e. the K-group of the reduced C^* algebra of functions $C_0(\mathbb{R}^{p+1}) = C(S^{p+1})$. Hence $K_1(\mathbb{R}^{p+1}) = K_1(S^{p+1})$. Their charges, constraining the dynamics of the brane, are dually described by $K^1(\mathbb{R}^{9-p}) = K^1(S^{9-p})$. In the case of D6-branes we have $K^1(S^3)$ as classifying the RR charges of flat D6-branes in flat 10-dimensional spacetime [41].

In the presence of a non-vanishing B-field for a stable D6-brane, the B-field need to be non-trivial on space \mathbb{R}^3 transverse to the world-volume, hence S^3 . The classification of D6-brane charges in IIA type superstring theory in flat space is then given by the twisted K-theory $K_H(S^3)$, which is $K^1(S^3, H) = \mathbb{Z}_k$, where $0 \neq [dB] = [H] = k \in H^3(S^3, \mathbb{Z})$. Hence the dynamics of D6-branes in type IIA superstring theory on flat spacetime is influenced by non-zero B-field.

Now we follow the philosophy present already implicitly in our previous work that the source for the non-trivial B-field on S^3 , hence $H \neq 0$, is due to the exoticness of the ambient \mathbb{R}^4 . The motivation is certainly the fact that the given exotic \mathbb{R}_H^4 corresponds to the non-trivial class $[H] \in H^3(S^3, \mathbb{Z})$ and conversely, where S^3 is taken from the boundary of the Akbulut cork [10, 9]. Moreover, exotic smoothness of \mathbb{R}_H^4 twists the K-theory groups $K^*(S^3)$ [8] provided S^3 lies at the boundary of the Akbulut cork.

Hence the possible dynamics (the charges) of D6-branes in spacetime $\mathbb{R}_H^4 \times \mathbb{R}^{5,1}$ is equivalently referred to the dynamics of D6-brane in the presence of non-zero B-field on transversal \mathbb{R}^3 .

Theorem 5. *RR charges of D6-branes in string theory IIA in the presence of non-trivial B-field ($[dB] \neq 0$), (these charges are classified by $K_H(S^3)$, $H \neq 0$ and $[H] \in H^3(S^3, \mathbb{Z})$), are related with exotic smoothness of small \mathbb{R}_H^4 . This exotic \mathbb{R}_H^4 corresponds to $[H]$ which twists $K(S^3)$ [8], where $S^3 \subset \mathbb{R}^4$ lies at the boundary of the Akbulut cork and S^3 is transverse to the branes. Thus, changing the smoothness of \mathbb{R}^4 gives rise to the change of the allowed charges for D6 branes, hence the dynamics changes.*

We see that geometric realization of (classical) D-branes in certain backgrounds of string theory is correlated with small exotic \mathbb{R}^4 's which can be all embedded in the standard smooth \mathbb{R}^4 . We saw in previous subsection 3.4 that quantum D-branes correspond to the net of exotic smooth \mathbb{R}^4 's embedded in certain exotic smooth \mathbb{R}^4 . Also an intriguing interpretation for this correspondence can be given: *in some limit of IIA superstring theory, small exotic smooth \mathbb{R}^4 's can be considered as carrying the RR charges of D6 branes.*

We will come back to these interesting points in the next section.

4. From wild embeddings to quantum D-branes

In this section we try to give a geometric approach to quantum D-branes using wild embeddings of trivial complexes into S^n or \mathbb{R}^n . Furthermore we are able to obtain a low-dimensional interpretation of D-brane charges. This point of view is supported by the Theorem 4 above. Here we will describe a dimension-independent way: every wild embedding j of a p -dimensional complex K into the n -dimensional sphere S^n is determined by the fundamental group $\pi_1(S^n \setminus j(K))$ of the complement. This group is perfect and uniquely representable by a 2-dimensional complex, a singular disk or grope (see [19]). As we showed in [10], the exotic \mathbb{R}^4 is given by the grope. Thus, every quantum D-brane must be determined (as a kind of germ) by some exotic \mathbb{R}^4 .

4.1. Wild and tame embeddings

We call a map $f : N \rightarrow M$ between two topological manifolds an embedding if N and $f(N) \subset M$ are homeomorphic to each other. From the differential-topological point of view, an embedding is a map $f : N \rightarrow M$ with injective differential on each point (an immersion) and N is diffeomorphic to $f(N) \subset M$. An embedding $i : N \hookrightarrow M$ is *tame* if $i(N)$ is represented by a finite polyhedron homeomorphic to N . Otherwise we call the embedding *wild*. There are famous wild embeddings like Alexanders horned sphere or Antoine's necklace. In physics one uses mostly tame embeddings but as Cannon mentioned in his overview [18], one needs wild embeddings to understand the tame one. As shown by us [10], wild embeddings are needed to understand exotic smoothness. As explained in [18] by Cannon, tameness is strongly connected to another topic: decomposition theory (see the book [24]).

Two embeddings $f, g : N \rightarrow M$ are said to be isotopic, if there exists a homeomorphism $F : M \times [0, 1] \rightarrow M \times [0, 1]$ such that

1. $F(y, 0) = (y, 0)$ for each $y \in M$ (i.e. $F(., 0) = id_M$)
2. $F(f(x), 1) = g(x)$ for each $x \in N$, and
3. $F(M \times \{t\}) = M \times \{t\}$ for each $t \in [0, 1]$.

If only the first conditions can be fulfilled then one call it concordance. Embeddings are usually classified by isotopy. An important example is the embedding $S^1 \rightarrow \mathbb{R}^3$, known as knot, where different knots are different isotopy classes.

4.2. Embeddings of $(4k - 1)$ - into $6k$ -manifolds

Now we start with a short discussion of embeddings $S^3 \rightarrow S^6$ as the example $k = 1$ of a general map $S^{4k-1} \rightarrow S^{6k}$. As Haefliger [30] showed the isotopy classes of embeddings are determined by the integer classes (Hopf invariant) in $H^3(S^3, \mathbb{Z})$. Thus the $4k - 1$ space is knotted in the $6k$ space. This phenomenon depends strongly on smoothness, i.e. it disappears for continuous or PL embeddings. Usually every n -sphere or every homology n -sphere unknots (in PL or TOP) in \mathbb{R}^m for $m \geq n + 3$, i.e. for codimension $m - n = 3$ or higher. Of course, one has the usual knotting phenomena in codimension 2 and the codimension 1 was shown to be unique for embeddings $S^n \rightarrow S^{n+1}$ (for $n \geq 6$) but is hard to solve in other cases.

Let $\Sigma \rightarrow S^6$ be an embedding of a homology 3-sphere Σ (containing the case S^3). Then the normal bundle of F is trivial (definition of an embedding) and homotopy classes of trivializations of the normal bundle (normal framing) are classified by the homotopy class $[\Sigma, SO(3)]$ with respect to some fixed framing. There is an isomorphism $[\Sigma, SO(3)] = [\Sigma, S^2]$ (so-called Pontrjagin-Thom construction) and $[\Sigma, S^2]$ can be identified with $H^3(\Sigma, \mathbb{Z}) = \mathbb{Z}$. That is one possible way to get the classification of isotopy classes of embeddings $\Sigma \rightarrow S^6$ by elements of $H^3(\Sigma, \mathbb{Z}) = \mathbb{Z}$. A class $[H]$ in $H^3(\Sigma, \mathbb{Z})$ determines via Stokes theorem

$$\int_{\Sigma=\partial A} H = \int_A dH$$

the 4-form dH in the 4-manifold¹ A with $\partial A = \Sigma$. As we know the (small) exotic \mathbb{R}^4 is determined by a contractable submanifold A , the Akbulut cork, with boundary ∂A a homology 3-sphere. The contractability of A implies $H^4(A, \mathbb{Z}) = 0$, i.e. every 4-form on A is given by dH for some 3-form H . The isomorphism $H^4(A, \partial A) = H^3(\partial A)$ and Stokes theorem imply

$$\int_A dH = \int_{\partial A} H = Q \neq 0$$

the non-vanishing of the 4-form $dH = Q \cdot d\text{vol}(A)$ with the volume form $d\text{vol}(A)$ of A normed to one. Combined with our result that $H^3(S^3, \mathbb{Z})$ determines some exotic \mathbb{R}^4 we have:

Theorem 6. *(The topological origins of the allowed D6-brane charges)*

Let \mathbb{R}_H^4 be some exotic \mathbb{R}^4 determined by some 3-form H , i.e. by a codimension-1 foliation on the boundary ∂A of the Akbulut cork A . The codimension-1 foliation on ∂A is determined by $H^3(\partial A, \mathbb{R})$. Each integer class in $H^3(\partial A, \mathbb{Z})$ determines the isotopy class of an embedding $\partial A \rightarrow S^6$. Hence, the group of allowed charges of D6-branes in the presence of B-field in M^{10} , i.e. $K_H^3(S^3)$ with $dB = H$, is determined equivalently by the isotopy classes of embeddings

¹To every 3-manifold Σ , there is a 4-manifold A with $\partial A = \Sigma$.

$\partial A \rightarrow S^6$. The classes of H -field are topologically determined by the isotopy classes of the embeddings, which affects the allowed charges of $D6$ -branes.

But more is true. Given two embeddings $F_i : \Sigma_i \rightarrow S^6$ between two homology 3-spheres Σ_i for $i = 0, 1$. A homology cobordism is a cobordism between Σ_0 and Σ_1 . This cobordism can be embedded in $S^6 \times [0, 1]$ determining the homology bordism class of the embedding. Then two embeddings of an oriented homology 3-sphere in S^6 are isotopic if and only if they are homology bordant.

4.3. Real cohomology classes and wild embeddings

Wild embeddings are important to understand usual embeddings. Consider a closed curve in the plane. By common sense, this curve divides the plane into an interior and an exterior area. The Jordan curve theorem agrees with that view completely. But what about one dimension higher, i.e. consider the embedding $S^2 \rightarrow \mathbb{R}^3$? Alexander was the first who constructed a counterexample, Alexanders horned sphere [3], as wild embedding $D^2 \rightarrow \mathbb{R}^3$. The main property of this wild object D_W^2 is the non-simple connected complement $\mathbb{R}^3 \setminus D_W^2$. In the following we will concentrate on wild embeddings of spheres S^n into spheres S^m equivalent to embeddings of \mathbb{R}^n into \mathbb{R}^m relative to the infinity ∞ point or to relative embeddings of D^n into D^m (relative to its boundary). From the physical point of view, D-branes or M-branes are topological objects of a trivial type like \mathbb{R}^n , S^n or D^n .

Lets start with the case of a finite k -dimensional polyhedron K^k (i.e. a piecewise-linear version of a k -disk D^k). Consider the wild embedding $i : K \rightarrow S^n$ with $0 \leq k \leq n - 3$ and $n \geq 7$. Then, as proofed in [26], the complement $S^n \setminus i(K)$ is non-simple connected with a countable generated (but not finitely presented) fundamental group $\pi_1(S^n \setminus i(K)) = \pi$. Furthermore, the group π is perfect (i.e. generated by the commutator subgroup $[\pi, \pi] = \pi$ implying $H_1(\pi) = 0$) and $H_2(\pi) = 0$ (π is called a superperfect group). With other words, π is a group where every element $x \in \pi$ can be generated by a commutator $x = [a, b] = aba^{-1}b^{-1}$ (including the trivial case $x = a$, $b = e$). By using geometric group theory, we can represent π by a grope (or generalized disk, see Cannon [19]), i.e. a hierarchical object with the same fundamental group as π (see below). In [10], the grope was used to construct a non-trivial involution of the 3-sphere connected with a codimension-1 foliation of the 3-sphere classified by the real cohomology classes $H^3(S^3, \mathbb{R})$. By using the suspension

$$\Sigma X = X \times [0, 1] / (X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times [0, 1])$$

of a topological space (X, x_0) with base point x_0 , we have an isomorphism of cohomology groups $H^n(S^n) = H^{n+1}(\Sigma S^n)$. Thus the class in $H^3(S^3, \mathbb{R})$ induces classes in $H^n(S^n, \mathbb{R})$ for $n > 3$ represented by a wild embedding $i : K \rightarrow S^n$ for some k -dimensional polyhedron. Then every small exotic \mathbb{R}^4 determines also higher brane charges:

Theorem 7. Let \mathbb{R}_H^4 be some exotic \mathbb{R}^4 determined by element in $H^3(S^3, \mathbb{R})$, i.e. by a codimension-1 foliation on the boundary ∂A of the Akbulut cork A .

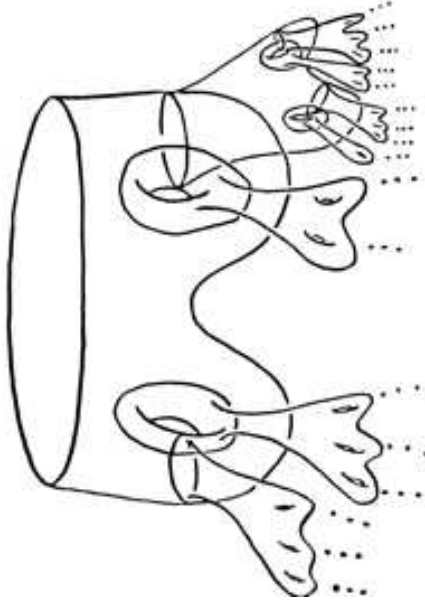


Figure 1: An example of a grope

Each wild embedding $i : K^3 \rightarrow S^p$ for $p > 6$ of a 3-dimensional polyhedron (as part of S^3) determines a class in $H^p(S^p, \mathbb{R})$ which can be interpreted as the charge of a Dp brane in the sense of Theorem 6.

4.4. C^* -algebras associated to wild embeddings

As described above, a wild embedding $j : K \rightarrow S^n$ of a polyhedron K is characterized by its complement $M(K, j) = S^n \setminus j(K)$ which is non-simple connected (i.e. the fundamental group $\pi_1(M(K, j))$ is non-trivial). The fundamental group $\pi_1(M(K, j)) = \pi$ of the complement $M(K, j)$ is a superperfect group, i.e. π is identical to its commutator subgroup $\pi = [\pi, \pi]$ (then $H_1(\pi) = 0$) and $H_2(\pi) = 0$. This group is not finite in case of a wild embedding. Here we use gropes to represent π geometrically. The idea behind that approach is very simple: the fundamental group of the 2-dimensional torus T^2 is the abelian group $\pi_1(T^2) = \langle a, b \mid [a, b] = aba^{-1}b^{-1} = e \rangle = \mathbb{Z} \oplus \mathbb{Z}$ generated by the two standard slopes a, b . The capped torus $T^2 \setminus D^2$ has an additional element c in the fundamental group generated by the boundary $\partial(T^2 \setminus D^2) = S^1$. This element is represented by the commutator $c = [a, b]$. In our superperfect group we have the same problem: every element c is generated by the commutator $[a, b]$ of two other elements a, b which are also represented by commutators etc. Thus one obtains a hierarchical object, a generalized 2-disk or a grope (see Fig. 1).

Now we describe two ways to associate a C^* -algebra to this grope. This first approach uses a combination of our previous papers [10, 8]. Then every

grope determines a codimension-1 foliation of the 3-sphere and vice versa. The leaf-space of this foliation is a factor III_1 von Neumann algebra and we have a C^* -algebra for the holonomy groupoid. For later usage, we need a more direct way to construct a C^* -algebra from a wild embedding or grope. The main ingredient is the superperfect group π , countable generated but not finitely presented group. To get an impression of this group, we consider a representation $\pi \rightarrow G$ in some infinite group. As the obvious example for G we choose the infinite union $GL(\mathbb{C}) = \bigcup_{\infty} GL(n, \mathbb{C})$ of complex, linear groups (induced from the embedding $GL(n, \mathbb{C}) \rightarrow GL(n+1, \mathbb{C})$ by an inductive limes process). Then we have a homomorphism

$$U : \pi \rightarrow GL(\mathbb{C})$$

mapping a commutator $[a, b] \in \pi$ to $U([a, b]) \in [GL(\mathbb{C}), GL(\mathbb{C})]$ into the commutator subgroup of $GL(\mathbb{C})$. But every element in π is generated by a commutator, i.e. we have

$$U : \pi \rightarrow [GL(\mathbb{C}), GL(\mathbb{C})]$$

and we are faced with the problem to determine this commutator subgroup. Actually, one has Whitehead's lemma (see [36]) which determines this subgroup to be the group of elementary matrices $E(\mathbb{C})$. One defines the elementary matrix $e_{ij}(a)$ in $E(n, \mathbb{C})$ to be the $(n \times n)$ matrix with 1's on the diagonal, with the complex number $a \in \mathbb{C}$ in the (i, j) -slot, and 0's elsewhere. Analogously, $E(\mathbb{C})$ is the infinite union $E(\mathbb{C}) = \bigcup_{\infty} E(n, \mathbb{C})$. Thus, every homomorphism descends to a homomorphism

$$U : \pi \rightarrow E(\mathbb{C}) = [GL(\mathbb{C}), GL(\mathbb{C})] \quad .$$

By using the relation

$$[e_{ij}(a), e_{jk}(b)] = e_{ij}(a)e_{jk}(b)e_{ij}(a)^{-1}e_{jk}(b)^{-1} = e_{ik}(ab) \quad i, j, k \text{ distinct}$$

one can split every element in $E(\mathbb{C})$ into a (group) commutator of two other elements.

Given a grope \mathcal{G} representing via $\pi_1(\mathcal{G}) = \pi$ the (superperfect) group π . Now we define the C^* -algebra $C^*(\mathcal{G}, \pi)$ associated to the grope \mathcal{G} with group π . The basic elements of this algebra are smooth half-densities with compact supports on \mathcal{G} , $f \in C_c^\infty(\mathcal{G}, \Omega^{1/2})$, where $\Omega_\gamma^{1/2}$ for $\gamma \in \pi$ is the one-dimensional complex vector space of maps from the exterior power $\Lambda^2 L$, of the union of levels L representing γ to \mathbb{C} such that

$$\rho(\lambda\nu) = |\lambda|^{1/2}\rho(\nu) \quad \forall \nu \in \Lambda^2 L, \lambda \in \mathbb{R}.$$

For $f, g \in C_c^\infty(\mathcal{G}, \Omega^{1/2})$, the convolution product $f * g$ is given by the equality

$$(f * g)(\gamma) = \int_{[\gamma_1, \gamma_2] = \gamma} f(\gamma_1)g(\gamma_2)$$

Then we define via $f^*(\gamma) = \overline{f(\gamma^{-1})}$ a $*$ -operation making $C_c^\infty(\mathcal{G}, \Omega^{1/2})$ into a $*$ -algebra. For each capped torus T in some level of the grope \mathcal{G} one has a natural

representation of $C_c^\infty(\mathcal{G}, \Omega^{1/2})$ on the L^2 space over T . Then one defines the representation

$$(\pi_x(f)\xi)(\gamma) = \int_{[\gamma_1, \gamma_2]=\gamma} f(\gamma_1)\xi(\gamma_2) \quad \forall \xi \in L^2(T).$$

The completion of $C_c^\infty(\mathcal{G}, \Omega^{1/2})$ with respect to the norm

$$\|f\| = \sup_{x \in M} \|\pi_x(f)\|$$

makes it into a C^* -algebra $C_c^\infty(\mathcal{G}, \pi)$. Via the representation $U : \pi \rightarrow E(\mathbb{C})$, we get a homomorphism into the usual convolution algebra $C^*(E(\mathbb{C}))$ of the group $E(\mathbb{C})$ used later to construct the action of the quantum D-brane. Finally we are able to define the C^* -algebra associated to the wild embedding:

Definition 3. *Let $j : K \rightarrow S^n$ be a wild embedding with $\pi = \pi_1(S^n \setminus j(K))$ as fundamental group of the complement $M(K, j) = S^n \setminus j(K)$. The C^* -algebra $C_c^\infty(K, j)$ associated to the wild embedding is defined to be $C_c^\infty(K, j) = C_c^\infty(\mathcal{G}, \pi)$ the C^* -algebra of the grope \mathcal{G} with group π .*

4.5. Isotopy classes of wild embeddings and KK theory

In section 4.1 we introduce the notion of isotopy classes for embeddings. Given two embeddings $f, g : N \rightarrow M$ with a special map $F : M \times [0, 1] \rightarrow M \times [0, 1]$ as deformation of f into g , then both embeddings are isotopic to each other. The definition is independent of the tameness or wilderness for the embedding. Now we specialize to our case of wild embeddings $f, g : K \rightarrow S^n$ with complements $M(K, f)$ and $M(K, g)$. The map $F : S^n \times [0, 1] \rightarrow S^n \times [0, 1]$ induces a homotopy of the complements $M(K, f) \simeq M(K, g)$ giving an isomorphism of the fundamental groups $\pi_1(M(K, g)) = \pi_1(M(K, f))$. Thus, the isotopy class of the wild embedding f is completely determined by the $M(K, f)$ up to homotopy. Using Connes work on operator algebras of foliation, our construction of the C^* -algebra for a wild embedding is functorial, i.e. an isotopy of the embeddings induces an isomorphism between the corresponding C^* -algebras. Given two non-isotopic, wild embeddings then we have a homomorphism between the C^* -algebras only. But every homomorphism (which is not a isomorphism) between C^* -algebras A, B gives an element of $KK(A, B)$ and vice verse. Thus,

Theorem 8. *Let $j : K \rightarrow S^n$ be a wild embedding with $\pi = \pi_1(S^n \setminus j(K))$ as fundamental group of the complement $M(K, j) = S^n \setminus j(K)$ and C^* -algebra $C_c^\infty(K, j)$. Given another wild embedding i with C^* -algebra $C_c^\infty(K, i)$. The elements of $KK(C_c^\infty(K, j), C_c^\infty(K, i))$ are the isotopy classes of the wild embedding j relative to i .*

4.6. Wild embeddings are quantum D-branes

Given a wild embedding $f : K \rightarrow S^n$ with C^* -algebra $C^*(K, f)$ and group $\pi = \pi_1(S^n \setminus f(K))$. In this section we will derive an action for this embedding to get back the D-brane action in the classical limit. The starting point is our remark above that the group π can be geometrically constructed by using a grope \mathcal{G} with $\pi = \pi_1(\mathcal{G})$. This grope was used to construct a codimension-1 foliation on the 3-sphere classified by the Godbillon-Vey invariant. This class can be seen as element of $H^3(BG, \mathbb{R})$ with the holonomy groupoid G of the foliation. The strong relation between the grope \mathcal{G} and the foliation gives an isomorphism for the C^* -algebra which can be easily verified by using the definitions of both algebras. As shown by Connes [22, 23], the Godbillon-Vey class GV can be expressed as cyclic cohomology class (the so-called flow of weights)

$$GV_{HC} \in HC^2(C_c^\infty(G)) \simeq HC^2(C_c^\infty(\mathcal{G}, \pi))$$

of the C^* -algebra for the foliation isomorphic to the C^* -algebra for the grope \mathcal{G} . Then we define an expression

$$S = Tr_\omega(GV_{HC})$$

uniquely associated to the wild embedding (Tr_ω is the Dixmier trace). S is the action of the embedding. Because of the invariance for the class GV_{HC} , the variation of S vanishes if the map f is a wild embedding. But this expression is not satisfactory and cannot be used to get the classical limit. For that purpose we consider the representation of the group π into the group $E(\mathbb{C})$ of elementary matrices. As mentioned above, π is countable generated and the generators can be arranged in the embeddings space. Then we obtain matrix-valued functions $X^\mu \in C_c^\infty(E(\mathbb{C}))$ as the image of the generators of π w.r.t. the representation $\pi \rightarrow E(\mathbb{C})$ labeled by the dimension $\mu = 1, \dots, n$ of the embedding space S^n . Via the representation $\iota : \pi \rightarrow E(\mathbb{C})$, we obtain a cyclic cocycle in $HC^2(C_c^\infty(E(\mathbb{C})))$ generated by a suitable Fredholm operator F . Here we use the standard choice $F = D|D|^{-1}$ with the Dirac operator D acting on functions $C_c^\infty(E(\mathbb{C}))$. Then the cocycle in $HC^2(C_c^\infty(E(\mathbb{C})))$ can be expressed by

$$\iota_* GV_{HC} = \eta_{\mu\nu} [F, X^\mu] [F, X^\nu]$$

using a metric $\eta_{\mu\nu}$ in S^n via the pull-back using the representation $\iota : \pi \rightarrow E(\mathbb{C})$. Finally we obtain the action

$$S = Tr_\omega([F, X^\mu] [F, X_\mu]) = Tr_\omega([D, X^\mu] [D, X_\mu] |D|^{-2}) \quad (12)$$

which can be evaluated by using the heat-kernel of the Dirac operator. For the classical limit, we take a tame embedding $f : K \rightarrow S^n$ of a p -dimensional complex K . Then the group π simplifies to a finite group or is trivial. The Dirac operator D on K acts on usual square-integrable functions and the action simplifies to

$$S = \int_K \left(\eta_{\mu\nu} \partial^\alpha X^\mu \partial_\alpha X^\nu + \frac{1}{3} R + \dots \right) dvol(K)$$

for the main contributions where R is the scalar curvature of K (for $p > 2$). It is known that this action agrees with the usual Born-Infeld action for p -branes ($p > 2$) if $R > 0$. Thus we obtain a description of the quantum D-brane action by using wild embeddings for the description of a quantum D-brane. We will further investigate this point in a forthcoming paper.

5. Conclusion

In this paper we present a lot of results to support our main conjecture:
The exotic small \mathbb{R}^4 lies at the heart of quantum gravity. Especially it is a quantized object.

Here we are mainly concentrated on the various relation to branes in superstring theory as a possible candidate of quantum gravity. We found the amazing connections between 4-exotics and NS and D-branes in various string backgrounds. We also studied the case of quantum D-branes using C^* -algebras. All the results can be simply summarized by:

The exotic small \mathbb{R}^4 as described by codimension-1 foliations on the 3-sphere is the germ of wide range of effects on D-branes. A quantum Dp -brane is given by a wild embedding of a p -dimensional complex into a n -dimensional space described by a two-dimensional complex, a grope.

Further evidences supported this statement as well the relation with supersymmetry and realistic QFT will be presented in a separate paper.

But as known from our previous work, the grope is the main structure to get the relation between the exotic small \mathbb{R}^4 and the codimension-1 foliation on the 3-sphere. The description of the wild embedding is rather independent of the dimensions ($n > 6$, $p > 2$). That is the reason why the exotic small \mathbb{R}^4 appeared in so different situations above!

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Appendix A. Remarks on K-matrix string theory and BSFT

D-branes are solitonic objects in string theory similar to instantons in gauge theory. Both can be seen as generically nonperturbative objects and one can consider dual theories on backgrounds of these. To investigate less supersymmetric phases of string theory one can introduce non-BPS branes. In fact in flat space background of type II string theory there are two kinds of D-branes systems without supersymmetry. They are brane-anti-brane systems and non-BPS D-branes. From the point of view of open string theory on such systems of branes the spectra of open strings contain tachyon's modes. In the closed

string spectra tachyons do not appear. The appearance of tachyons is the indication that the theory (without supersymmetry) is in its unstable phase and will decay into the stable phase where some supersymmetry is restored. The conjecture by Ashoke Sen states that the unstable D-branes decay to the closed backgrounds without any brane or to stable D-branes, without tachyons. This decay process is called tachyon condensation, and is a dynamical phenomena which can not be described in perturbative open string theory since we relate two different backgrounds. Similarly as in various dualities in string theory where different backgrounds are related, also here we deal rather with important non-perturbative aspects of the theory. The Ashok conjecture one can check by techniques of noncommutative field theory, K-theory and field string theory. This last includes boundary field string theory (BFST) as introduced by Witten where one can calculate exact expressions for the tachyon potential and effective actions for the systems of unstable D-branes (including DBI-like and WZ ones).

In fact any kind of D-brane in type II superstring theory can be obtained as a soliton in the gauge theory defined on higher dimensional unstable D-brane systems. The gauge theory is equipped with a tachyon field. From this descent relation, from unstable D-branes to the lower dimensional solitons - D-branes, follows, in particular, the classification of the RR charges of D-branes in terms of K-theory. There is, however, yet another, ascent way to D-branes: these are realized as bound states of lower dimensional unstable D-brane systems. In a particular case of string theory described by Matrix theory and lowest possible dimension of unstable D-branes, namely D-instantons (D-particles), the construction of type II D-branes as bound states of D-instantons is possible and this is what one calls K-matrix theory. The reason is particularly close and natural connection of K-homology classes (and spectral triples from non-commutative geometry) with world-volumes of D-branes as we discussed in this paper. Moreover, the ascent and descent relations together give rise to the same D-branes hence the suitable K-theory describing D-branes should be enhanced to the KK-theory of Kasparov in the context of C^* algebra bi-modules. The tachyon condensation is crucial for both, ascent and descent relations leading to D-branes and for the D-instantons case in K-matrix string theory.

Following [6] let us see how tachyons and other fields on unstable brane systems emerge from BSFT description. The boundary states ascribed to a D-brane are linear combinations of the states $|Bp; \pm \rangle$, which are represented as the formal integration out of longitudinal fields along the world-volume, i.e.

$$|Bp; \pm \rangle = \int [x^\alpha][d\psi^\alpha] |X^\alpha, x^i = 0 \rangle |\psi^\alpha, \psi^i = 0; \pm \rangle$$

where α are longitudinal and i transversal directions to the Dp -brane. Turning on the fields on the world-volume is via the boundary interactions and modifies the boundary states as

$|Bp; \pm \rangle_b = \int [x^\alpha][d\psi^\alpha] e^{-S_b(x, \psi)} |X^\alpha, x^i = 0 \rangle |\psi^\alpha, \psi^i = 0; \pm \rangle$ where S_b is the boundary action of boundary interactions. This can be seen in the case of

gauge fields on the brane as supersymmetric generalization of operators represented by Wilson loops

$e^{-S_b(X, \Psi_{\pm})} = \text{TrP} \exp \left\{ - \int d\sigma \left(A_{\alpha}(X) \dot{X}^{\alpha} \right) - \frac{1}{2} F_{\alpha\beta}(X) \Psi_{\pm}^{\alpha} \Psi_{\pm}^{\beta} \right\}$. Rewriting this in terms of superfields $\hat{X}(\sigma, \theta) = X^{\mu}(\sigma) + i\theta \Psi_{\pm}^{\mu}(\sigma)$, $\hat{x}^{\mu}(\sigma, \theta) = x^{\mu}(\sigma) + i\theta \sigma_{\pm}^{\mu}(\sigma)$ and the covariant superderivative, $D = \partial_{\theta} + \theta \partial_{\sigma}$, where (σ, θ) are super coordinates on the boundary, we get

$$e^{-S_b(X, \Psi_{\pm})} = \text{Tr} \tilde{P} \exp \left(- \int d\sigma d\theta \left(A_{\alpha}(\hat{X}) \right) D \hat{X}^{\alpha} \right)$$

where \tilde{P} is the supersymmetric path-ordered product.

Now the boundary action can be given for the case of unstable Dp-branes (IIA type) or the system of non BPS branes - antibranes (type IIB). We introduce the matrix

$$\hat{M} = \begin{pmatrix} -A_{\alpha}(\hat{X}) D \hat{X}^{\alpha} - i\Phi^i(\hat{X}) \hat{P}_i & T(\hat{X}) \\ T(\hat{X})^{\dagger} & -A'_{\alpha}(\hat{X}) D \hat{X}^{\alpha} - i\Phi'^i(\hat{X}) \hat{P}_i \end{pmatrix}$$

here $\hat{P}_i = \theta P_i(\sigma) + i\Pi_{i\pm}(\sigma)$ and $P_i, \Pi_{i\pm}$ are conjugate momenta of $X_i, \Psi_{i\pm}$ respectively. $A_{\alpha}, A'_{\alpha}, \Phi^i, \Phi'^i, T$ are gauge fields, scalar fields and tachyon on the brane and anti brane which in the case of non-BPS Dp-brane are not independent, i.e. $A_{\alpha} = A'_{\alpha}$ and $\Phi^i = \Phi'^i$. Matrix \hat{M} is rewritten in terms of Pauli matrices σ_1, σ_2 and in terms of redefined fields: $A_{\alpha}^{\pm} = \frac{1}{2}(A_{\alpha} \pm A'_{\alpha})$, $\Phi_{\alpha}^{\pm} = \frac{1}{2}(\Phi_{\alpha} \pm \Phi'_{\alpha})$, $T^{\pm} = \frac{1}{2}(T \pm T^{\dagger})$ as:

$$\hat{M} = -(A_{\alpha}^{+} D \hat{X}^{\alpha} + i\Phi_{+}^i P_i) \otimes 1_2 - (A_{\alpha}^{-} D \hat{X}^{\alpha} + i\Phi_{-}^i P_i) \otimes \sigma_2 \sigma_1 + T^{+} \sigma_1 + T^{-} \sigma_1.$$

Introducing additional real fermionic superfields

$$\hat{\Gamma}^I(\sigma, \theta) = \eta^I(\sigma) + \theta F^I(\sigma) \quad (\text{A.1})$$

$I = 1, 2$, we obtain, now for the case of N non-BPS pairs of Dp - anti-Dp branes, the boundary action:

$$e^{-S_b} = \int [d\hat{\Gamma}^1][d\hat{\Gamma}^2] \text{Tr} \tilde{P} \exp \left\{ \int d\sigma d\theta \left(\frac{1}{4} \hat{\Gamma}^1 D \hat{\Gamma}^1 + \frac{1}{4} \hat{\Gamma}^2 D \hat{\Gamma}^2 + (A_{\alpha}^{+} D \hat{X}^{\alpha} + i\Phi_{+}^i P_i) + (A_{\alpha}^{-} D \hat{X}^{\alpha} + i\Phi_{-}^i P_i) \hat{\Gamma}^2 \hat{\Gamma}^1 + T^{+} \hat{\Gamma}^1 + T^{-} \hat{\Gamma}^2 \right) \right\}.$$

Next one expands \hat{M} in terms of $SO(2m)$ gamma matrices, namely the antisymmetrized product $\Gamma^{I_1 \dots I_k} = \Gamma^{[I_1} \dots \Gamma^{I_k]}$

$$\hat{M} = \sum_{k=0}^{2m} \hat{M}^{I_1 \dots I_k} \otimes \Gamma^{I_1 \dots I_k}.$$

The form of the boundary action follows:

$$e^{-S_b} = \int [d\hat{\Gamma}^I] \text{Tr} \tilde{P} \exp \left\{ \int d\sigma d\theta \left(\frac{1}{4} \hat{\Gamma}^I D \hat{\Gamma}^I + \sum_{k=0}^{2m} \hat{M}^{I_1 \dots I_k} \otimes \hat{\Gamma}^{I_1 \dots I_k} \right) \right\}.$$

The θ integration can be performed which after integrating out the F fields in (A.1), gives

$$e^{-S_b} = \int [d\eta^I] \text{Tr} \tilde{P} \exp \left\{ \int d\sigma \left(\frac{1}{4} \eta^I \dot{\eta}^I + \sum_{k=0}^{2m} M^{I_1 \dots I_k} \otimes \Gamma^{I_1 \dots I_k} \right) \right\}.$$

This last path integral expression is equivalent to the operator one, after integrating on $d\eta$ and in the superfields formalism the action reads:

$$e^{-S_b} = \begin{cases} \text{Tr} \tilde{P} e^{\int d\sigma d\theta \hat{M}(\sigma)}, & \text{for Dp - anti - Dp - branes} \\ \frac{1}{\sqrt{2}} \text{Tr} \tilde{P} e^{\int d\sigma d\theta \hat{M}(\sigma)}, & \text{for non - stable Dp - branes} \end{cases} \quad (\text{A.2})$$

which in terms of fields is (in the NS sector again):

$$e^{-S_b} = \begin{cases} \text{Tr} P e^{\int d\sigma M(\sigma)}, & \text{for Dp - anti - Dp - branes} \\ \frac{1}{\sqrt{2}} \text{Tr} P e^{\int d\sigma M(\sigma)}, & \text{for non - stable Dp - branes} \end{cases} \quad (\text{A.3})$$

Turning to the D-instantons case let us consider N non BPS D-instantons which are the lowest dimensional D-branes in type IIA theory. The gauge theory on such systems is $U(N)$ gauge theory. No gauge field A are present but bosons of the theory consist of 10 scalar fields Φ^μ , $\mu = 0, 1, \dots, 9$ and tachyon T which are from the adjoint representation of $U(N)$. One takes limit $N \rightarrow \infty$ since creation or annihilation of arbitrary many of non-BPS D-instantons can be considered then. Thus the infinite Hermitian matrices Φ^μ , T represent the linear operators acting on the separable Hilbert space \mathcal{H} . The BSFT action for such system for the NSNS sector, reads

$$S(\Phi^\mu, T) = \frac{2\pi}{g_s} \langle 0 | e^{-S_b(\Phi^\mu, T)} | B(-1); + \rangle_{NS} \quad (\text{A.4})$$

Here $S_b(\Phi^\mu, T)$ is the action governing the interaction of the boundary states from closed strings Hilbert space \mathcal{H} as before. The matrix \hat{M} in (A.2) is now given as

$$\begin{pmatrix} -i\Phi^\mu \hat{P}_\mu & T \\ T & -i\Phi^\mu \hat{P}_\mu \end{pmatrix}$$

or M in (A.3), as

$$\begin{pmatrix} -i\Phi^\mu P_\mu - T^2 - \frac{1}{2}[\Phi^\mu, \Phi^\nu] \Pi_\mu \Pi_\nu & -[\Phi^\mu, T] \Pi_\mu \\ -[\Phi^\mu, T] \Pi_\mu & -i\Phi^\mu P_\mu - T^2 - \frac{1}{2}[\Phi^\mu, \Phi^\nu] \Pi_\mu \Pi_\nu \end{pmatrix}.$$

The very important thing is that any D-brane configuration of IIA string theory can be constructed in the above K-matrix string theory from D-instantons. Thus, solution representing Dp-brane is the following configuration:

$$T = u \sum_{\alpha=0}^p \tilde{p}_\alpha \otimes \gamma^\alpha, \quad \Phi^\alpha = \tilde{x}^\alpha \otimes 1, \quad \alpha = 0, \dots, p, \quad \Phi^i = 0, \quad i = p+1, \dots, 9 \quad (\text{A.5})$$

which becomes an exact solution in the limit $u \rightarrow \infty$, and \tilde{p} , \tilde{x} are operators acting on \mathcal{H} . The eigenvalues of Φ^μ represent the position of non-BPS D-instantons and Dp-brane extends over $0, \dots, p$ directions. In the case of N Dp-branes one has:

$$T = u \sum_{\alpha=0}^p \tilde{p}_\alpha \otimes 1_N \otimes \gamma^\alpha, \quad \Phi^\alpha = \tilde{x}^\alpha \otimes 1, \quad \alpha = 0, \dots, p, \quad \Phi^i = 0$$

the Hilbert space for T and Φ^μ is $\mathcal{H} \otimes \mathbb{C}^N \otimes S$ and S is spinor space representing γ^α . The gauge fields corresponding to the $U(N)$ symmetry, which is now a field on Dp-brane world-volume, reappear and modify the tachyon operator as

$$T = u \sum_{\alpha=0}^p (\tilde{p}_\alpha \otimes 1_N - i A_\alpha(\tilde{x})) \otimes \gamma^\alpha. \quad (\text{A.6})$$

We see the close connection of tachyon and Dirac operators.

Appendix B. Tachyons and D brane charges

The Chern-Simons term in BSFT is obtained by considering the state $\langle C|$ in closed string theories corresponding to the RR field C , which couples to the boundary state $|Bp; + \rangle$ representing the Dp-brane, i.e.

$$S_{CS}(C, T, A_\alpha, \Phi^i) = \langle C | e^{-S_b} | Bp; + \rangle_{RR}$$

Again, following [6], the relation between non-stable D(-1)-branes (D-instantons) in K-string theory with Dp-branes in type II seen from BSFT formalism, allows to state the equality:

$$S_{CS}^{D(-1)}(C, T, \Phi^\mu) = S_{CS}^{Dp}(C, A_\mu, t, \phi^i) \quad (\text{B.1})$$

where A_μ, t, ϕ^i are rather fields on the world-volume of the Dp-brane than operators on Hilbert space as on the lhs. of (B.1). This equality of couplings to RR fields, i.e. Chern-Simons terms, one from operator side and the second as in gauge theory, gives rise to a deep relation which illustrates the Atiyah-Singer index theorem. Let us note that when Dp-brane is BPS the tachyon field t is absent and when we additionally nullify the scalar fields ϕ^i , the CS term becomes usual one as in gauge theory, i.e. $S_{CS}^{Dp}(C, A_\mu) = \mu_p \int_{Dp} C \wedge \text{tr} e^{2\pi \mathcal{F}}$.

The Chern-Simons term for the non-BPS D-instantons is given as ([6], p. 27):

$$S_{CS}^{D(-1)} = \langle \psi_2^\mu = 0 | d^{10} k C(k^\mu, \psi_1^\mu) \text{Str}(e^{-ik_\mu \Phi^\mu + 2\pi \mathcal{F}}) | \psi_1^\mu = 0 \rangle \quad (\text{B.2})$$

where Str is the supersymmetrized trace and

$$\mathcal{F} = -T^2 + \frac{1}{8\pi^2} [\Phi^\mu, \Phi^\nu] \psi_2^\mu \psi_1^\nu - \frac{i}{2\pi} [\Phi^\mu, T] \psi_2^\mu \sigma_1.$$

The similar CS term for the pairs of D-instantons - anti-D-instantons from IIB type K-matrix string theory, reads ([6], p. 28-29).

$$S_{CS}^{D(-1)} = \langle \psi_2^\mu = 0 | d^{10} k C(k^\mu, \psi_1^\mu) \text{Str}(e^{-ik_\mu \tilde{\Phi}^\mu + 2\pi \tilde{\mathcal{F}}}) | \psi_1^\mu = 0 \rangle \quad (\text{B.3})$$

here

$$\tilde{\mathcal{F}} = \begin{pmatrix} -TT^\dagger + \frac{1}{8\pi^2} [\Phi^\mu, \Phi^\nu] \psi_2^\mu \psi_2^\mu & -\frac{i}{2\pi} (\Phi^\mu T - T \Phi'^\mu) \psi_2^\mu \\ -\frac{i}{2\pi} (\Phi'^\mu T^\dagger - T^\dagger \Phi^\mu) \psi_2^\mu & -T^\dagger T + \frac{1}{8\pi^2} [\Phi^\mu, \Phi^\nu] \psi_2^\mu \psi_2^\mu \end{pmatrix} \quad (\text{B.4})$$

Supposing $k_\mu = 0$ and all components but C_0 vanish, the scalar fields Φ^μ , Φ'^μ vanish too. Thus we have:

$$S_{CS}^{D(-1)} = C_0 \text{Str} \exp \left(-2\pi \begin{pmatrix} 0 & T \\ T^\dagger & 0 \end{pmatrix}^2 \right) = C_0 \text{Ind} \left(\begin{pmatrix} 0 & T \\ T^\dagger & 0 \end{pmatrix} \right)$$

which means $S_{CS}^{D(-1)} = C_0 (\dim \ker TT^\dagger - \dim \ker T^\dagger T)$, i.e. the index of the tachyon operator is interpreted as D-instanton charge.

When BPS Dp-brane is present the tachyon operator, following (A.6), is

$$u \sum_{\alpha=0}^p (\tilde{p}^\alpha - i A_\alpha(\tilde{x})) \Gamma^\alpha = -iuD$$

where D is the Dirac operator and Γ^α are $p+1$ $SO(p+1)$ gamma matrices for odd p . We see that the D(-1) charge in the presence of Dp-brane is just the index of the Dirac operator defined on the world-volume of Dp-brane.

From the equality (B.1) of CS terms and the coupling $S_{CS}^{Dp}(C, A_\mu) = \mu_p \int_{Dp} C \wedge \text{tr} e^{2\pi F}$, we have

$$\text{index}(-iD) = \int_{Dp} \text{tr} e^{F/2\pi}$$

which is nothing but the variant of the Atiyah-Singer index theorem. Thus ([6], p. 30), *the Dirac operator is the tachyon operator representing the Dp-brane in the system of D-instanton - anti-D-instanton. The index of the Dirac operator is the charge of D-instantons. The D-instanton can be equivalently described as the ordinary instantons configuration in the gauge theory defined on the world-volume of Dp-brane. In that case the Chern number of the gauge bundle on the world-volume is the instanton number.*

The above observation opens the possibility to interpret D-branes as spectral triples also in more general non-commutative situations. The appearance of the Atiyah-Singer theorem in the context of charges of Dp-branes can be understood also quite generally via relation of the charges with K-theory classes and the duality of K-theory with K-homology [34]. Namely let \mathcal{T} be a bounded linear operator acting on a separable Hilbert space \mathcal{H} with kernel and cokernel being finite dimensional vector spaces, i.e. a Fredholm operator. The index of it is

$$\text{index } \mathcal{T} = \dim \ker \mathcal{T} - \dim \text{coker } \mathcal{T} \quad (\text{B.5})$$

Let \mathcal{F} be the space of Fredholm operators on \mathcal{H} with the norm (of operators) topology. From (B.5) a continuous map $\text{index} : \mathcal{F} \rightarrow \mathbb{Z}$ is defined which is in fact a bijection between connected components of \mathcal{F} and \mathbb{Z} . For given X a compact topological space the set $[X, \mathcal{F}]$ of homotopy classes of maps $X \rightarrow \mathcal{F}$ with its monoid structure is isomorphic to K-theory classes:

$$[X, \mathcal{F}] \simeq K(X) \quad (\text{B.6})$$

This is basically since the kernel and cokernel of the continuous family of the Fredholm operators on X acting on \mathcal{H} , are vector bundles over X . Thus the K-theory class of the above pair of vector bundles derived from the above family of Fredholm operators, is determined. Let denote it as $\text{Index } \mathcal{T} := [(\ker \mathcal{T}, \text{coker } \mathcal{T})] \in K(X)$ [34].

For a point pt we have $K(pt) = \mathbb{Z}$ and the Index defined via K-theory of the family of Fredholm operators (B.6) becomes exactly the index of the Fredholm operator (B.5). The K-homology group $K_0(X)$, which is the set of homotopy classes of Fredholm operators $\{[\mathcal{P}]\}$, is dual to the K-theory and the duality is given by the pairing

$$([E], [\mathcal{P}]) \rightarrow \text{index}(P_E) \in \mathbb{Z}. \quad (\text{B.7})$$

Here E is the vector bundle $E \rightarrow X$ representing the class $[E] \in K(X)$ and $\mathcal{P} : \Gamma(X, E) \rightarrow \Gamma(X, E)$ is the Fredholm operator defined on the Hilbert space $\mathcal{H} = L^2(\Gamma(X, E))$. Let us observe now that Dirac operator defined on a spin manifold with vector bundle fits to this schema since it is Fredholm on a suitable Hilbert space [34]. Namely $iD : \Gamma(X, S_E^+) \rightarrow \Gamma(X, S_E^-)$ where $E \rightarrow X$ is a real spin bundle and S_E^\pm are twisted by E spinor bundles. Now the Chern character is defined for K-theory $\text{ch} : K(X) \rightarrow H^*(X, \mathbb{Q})$, taking it to the cohomology classes. In the case of smooth manifold and a smooth vector bundle $E \rightarrow X$ with a Hermitian connection with the curvature $\nabla_E^2 = F_E$, the Chern character reads

$$\text{ch}(E) = \text{tr } e^{F_E/2\pi i}.$$

We can determine the analytic index of iD via K-theory as before. Then using the Chern character (and Gysin map) one translates the analytic index to the topological one, via cohomology classes, which is just the Atiyah-Singer index theorem [34, 40]

$$\text{index}(-iD) = \int_X \text{ch}(E) \wedge \hat{A}(TX).$$

In terms of K-theory there is a natural bi-linear pairing given by the index of the twisted Dirac operator associated to the spin^c structure on X (E, F being vector bundles representing the K-theory classes):

$$(E, F)_K = \text{index}(D_{E \otimes F}). \quad (\text{B.8})$$

The Chern character gives rise to the ring isomorphism

$$\text{ch} : K(X) \otimes \mathbb{Q} \simeq H(X, \mathbb{Q})$$

which should be modified as $\text{ch} \rightarrow \sqrt{\text{Td}(X)} \smile \text{ch}$ in order to preserve (B.8). Here Td is the Todd (reversible) class of the tangent bundle of X . The above modification is again nothing but the Atiyah-Singer index theorem [15]:

$$\text{index}(D_{E \otimes F}) = C \int_X \text{ch}(E \otimes F) \smile \text{Td}(X).$$

The above duality and generalization of index theorem can be extended over noncommutative spaces given by C^* -algebras leading to the suitable formula for the charge of noncommutative branes [15, 14, 39].

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